Math 6321 – Spring 2013 Name:

Final Exam

Please, write clearly and justify all your steps to get proper credit for your work. You are supposed to work ON YOUR OWN and cannot discuss the problems with anyone else.

You are allowed to cite theorems/propositions proved in the lectures or in the textbook, but not statements of textbook problems or HW problems. Wording of the type "like in class" or "like in the HW" are not allowed to replace proofs or part of proofs.

(1) [8 Pts] Solve problems 48-49 from Ch.13 of textbook (p.332-333).

(2) [6 Pts] Recall that X^{-1} is the tempered distribution

$$X^{-1}[\phi] = P.V. \int \frac{\phi(x)}{x}\, dx, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

Prove that $(X^{-1})^\wedge(\xi) = -i\pi \operatorname{sgn}(\xi)$.

I suggest to proceed as follows.

(a) Let $f_b(x) = \begin{cases} -e^{bx} & \text{if } x < 0 \\ e^{-bx} & \text{if } x > 0 \end{cases}$ and show that f_b converges to $f = \operatorname{sgn}$ as $b \to 0$ in the sense of tempered distributions, that is, $f_b[\phi] \to \operatorname{sgn}[\phi]$ as $b \to 0$, for all $\phi \in \mathcal{S}$.

(b) Show that if $(F_k), F \in \mathcal{S}'$ and $F_k[\phi] \to F[\phi]$ for all $\phi \in \mathcal{S}$, then $\hat{F}_k[\phi] \to \hat{F}[\phi]$ for all $\phi \in \mathcal{S}$.

(c) Use (a) and (b) to derive that $(\operatorname{sgn})^\wedge = -2iX^{-1}$. You will then obtain the desired expression of $(X^{-1})^\wedge$ by using the Fourier inversion formula.

Comment (nothing to prove here): Note that this result is related to Prob. (1). In fact, using the convolution theorem we have that, for $f \in \mathcal{S}(\mathbb{R})$,

$$Hf(x) = \frac{1}{\pi} X^{-1} * f(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int \frac{f(x-y)}{y} dy.$$

Indeed this last observation can be extended to the case where $f \in L^2$. This however requires some technicalities to justify the convolution theorem.

(3) [4 Pts] Let $f \in L^1(\mathbb{R})$ and suppose that there exists $C > 0$ and $0 < \alpha < 1$ such that, for every $\xi \in \mathbb{R}$,

$$|\hat{f}(\xi)| \leq C \frac{1}{|\xi|^{1+\alpha}}.$$

Prove that there is a constant $K > 0$ such that the function f satisfies

$$|f(x+h) - f(x)| \leq K\,|h|^\alpha, \quad \text{for all } x, h \in \mathbb{R}.$$

(Hint: Start by expressing $f(x+h)-f(x)$ in terms of its Fourier transform.)

(4) [4 Pts] Let $f \in L^1[0,1]$ and set

$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt, \quad 0 < x \leq 1.$$

Show that g is defined a.e., $g \in L^1[0,1]$, and $\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx$.

Note: You need to verify the measurability of g.

(5) [Extra Credit Problem: 3 Pt]

Let $\{f_n : n \in \mathbb{N}\}$ be an orthonormal sequence in $L^2[a,b]$. Prove that $\{f_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2[a,b]$ if and only if

$$\sum_{n=1}^{\infty} \left| \int_a^x f_n(t)\,dt \right|^2 = x - a, \quad x \in [a,b].$$