Math 6321 – Spring 2013 Name:

HW #2

Please, write clearly and justify all your steps, to get proper credit for your work.

(1) [4 Pts] Let $1 \leq p < q \leq \infty$. Prove that $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\| = \|f\|_p + \|f\|_q.$$

(2) [6 Pts] Fix $1 \leq p < \infty$.

(a) Let S be the set of all *really simple functions*, that is, the real-valued functions of the form

$$\sum_{k=1}^{N} c_k \, \chi_{[a_k,b_k]},$$

where $c_k \in \mathbb{R}$ and $[a_k, b_k]$ are intervals in $\mathbb{R}$. Prove that S is dense in $L^p(\mathbb{R})$.

(b) Prove that $L^p(\mathbb{R})$ is *separable*, that is, it contains a countable dense subset. (Hint: use part (a) combined with the observation that you can choose an appropriate countable subset of S).

(c) Show that $L^\infty(\mathbb{R})$ is not separable. In fact, you can show that for any Lebesgue measurable subset $E \subset \mathbb{R}$, then $L^\infty(E)$ is not separable unless $\lambda(E) = 0$.

(3) [4 Pts] Solve Problem 24 from Chapter 10.