HW #1 SOLUTION (SKETCH)

1.2.4 $\ell_{t,\omega} = \{x \in \mathbb{R}^2 : \langle x, \omega\rangle = t, \quad t \in \mathbb{R}, \ \omega \in S^1\}$

$|t| = |\langle x,\omega\rangle| \le \|x\| \, \|\omega\| = \|x\|$

If $x = t\omega$, then $\langle t\omega, \omega\rangle = t$ and $\|x\| = |t|$

Here $|t| = \min\{x : x \in \ell_{t,\omega}\}$

1.2.10 Given $\varepsilon > 0$, we want to determine the existence of $\delta(\varepsilon, x)$ s.t. $|h_D(\theta) - h_D(\theta')| < \varepsilon$ if $|\theta - \theta'| < \delta$.

For a convex region D, let $\ell_{h_D(\theta), \omega(\theta)}$ and $\ell_{h_D(\theta'), \omega(\theta')}$ be two lines tangent to the boundary of D. We assume $|\theta - \theta'| \le \pi$.

By the convexity of D, both lines must be external to D, they have a point of ~~If $\omega(\theta) = \omega(\theta')$, then~~ intersection $x \in \mathbb{R}^2$ where the following eq. hold:

$$h_D(\theta) - h_D(\theta') = \langle x, \omega(\theta)\rangle - \langle x, \omega(\theta')\rangle$$

Then:

$$|h_D(\theta) - h_D(\theta')| = |x_1(\cos\theta - \cos\theta') + x_2(\sin\theta - \sin\theta')| \le$$

$$\le |x_1|\,|\cos\theta - \cos\theta'| + |x_2|\,|\sin\theta - \sin\theta'|$$

Due to the continuity of $\cos\theta$, $\sin\theta$, we can find $\delta(x, \varepsilon)$ s.t. $|h_D(\theta) - h_D(\theta')| < \varepsilon$ if $|\theta - \theta'| < \delta$.

1.2.12 To derive $h'_D(\theta) - s(\theta) = 0$, we used (1.29) and assumed the existence of an unique point of tangency, which is true only if D is strictly convex. Note that we need this condition to be able to differentiate $s(\theta)$ and derive 1.29

2.1.5 Since a is invertible, $ax = y$ iff $x = a^{-1}y$, x, y are in 1-1 correspondence

$$\mu = \min_{x \ne 0} \frac{\|ax\|}{\|x\|} = \min_{y \ne 0} \frac{\|y\|}{\|a^{-1}y\|} = \frac{1}{\max_{y \ne 0} \frac{\|a^{-1}y\|}{\|y\|}} = \frac{1}{|||a^{-1}|||}$$

2.1.6 Using 2.1.5: $C_a = |||a||| \ |||a^{-1}||| = \max_{x \ne 0} \frac{\|ax\|}{\|x\|} \Big/ \min_{x \ne 0} \frac{\|ax\|}{\|x\|}$

2.2.2 $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$ IS a NORM on $C[0,1]$

(proof omitted → STANDARD)

CONTINUITY Let $(f_k) \subset C[0,1]$ and suppose $\lim_{k \to \infty} \|f - f_k\|_\infty = 0$

~~Let (x_n) be a seq. of numbers in $[0,1]$ s.t. $\lim_n x_n = x_0$~~

For $x, y \in [0,1]$, $|f(x) - f(y)| \le |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$

Since f_k is continuous, given $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon)$ s.t. $|f_k(x) - f_k(y)| < \varepsilon/3$, whenever $|x - y| < \delta$, $\forall x, y \in [0,1]$

Since $f_k \to f$ uniformly, for the same $\varepsilon > 0$, $\exists\ N(\varepsilon)$ s.t. $|f_k(x) - f(x)| < \varepsilon/3$, if $k > N$.

Thus, $|f(x) - f(y)| < \varepsilon$, whenever $|x - y| < \delta$, for any $x, y \in [0,1]$