HW #2 SOLUTION

3.2.1 Recall that $\ell_{t,\omega(\theta)} = \ell_{-t,-\omega(\theta)} = \ell_{-t,\omega(\theta+\pi)}$

This implies that $Rf(t,\omega(\theta)) = Rf(-t,\omega(\theta+\pi))$

3.4.2 Let $f = \chi_{B_1(0)}$, where $B_1(0)$ is a ball of radius 1 and center 0

Denote by $f_a(x) = f(x-a)$. then $f_a = \chi_{B_1(a)}$.

For $a \in \mathbb{R}^2$, we can write $a = \langle a,\omega\rangle\omega + \langle a,\omega^\perp\rangle\omega^\perp$

$$Rf_a(\omega,t) = \int_{\mathbb{R}} f_a(s\omega^\perp + t\omega)\,ds = \int_{\mathbb{R}} f(s\omega^\perp + t\omega - a)\,ds$$

$$= \int_{\mathbb{R}} f\left((s-\langle a,\omega^\perp\rangle)\omega^\perp + (t-\langle a,\omega\rangle)\omega\right)ds$$

use $\sigma = s - \langle a,\omega^\perp\rangle$
$d\sigma = ds$

$$= \int_{\mathbb{R}} f(\sigma\omega^\perp + (t-\langle a,\omega\rangle)\omega)\,d\sigma$$

$$= Rf(\omega, t-\langle a,\omega\rangle)$$

Recall the fact that $Rf(\omega,t) = \begin{cases} 2\sqrt{1-t^2} & |t|\le 1 \\ 0 & |t|>1 \end{cases}$

Thus:

$$Rf_a(\omega,t) = \begin{cases} 2\sqrt{1-(t-\langle a,\omega\rangle)^2} & |t-\langle a,\omega\rangle|\le 1 \\ 0 & |t-\langle a,\omega\rangle|>1 \end{cases}$$

You can write $\langle a,\omega\rangle = a_1\cos\theta + a_2\sin\theta$, where $a = (a_1, a_2)$

3.4.6

$$\int_{\mathbb{R}} Rf(s,\omega)\,\varphi(t-s)\,ds = \int_{\mathbb{R}} \varphi(t-s)\int_{\mathbb{R}} f(\sigma\omega^\perp + s\omega)\,d\sigma\,ds$$

Note that $\varphi \in C_c(\mathbb{R})$ (continuous with compact support) and $f \in L^1(\mathbb{R}^2)$ and

Hence, by Fubini's theorem, $G(s,\sigma) = f(\sigma\omega^\perp + s\omega)\,\varphi(t-s) \in L^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}} Rf(s,\omega)\,\varphi(t-s)\,ds = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t-s)\,f(\sigma\omega^\perp + s\omega)\,ds\,d\sigma = 0$$

since $G(s,\sigma) = 0$ except on a set of measure 0

Implication: The convolution of Rf and a continuous filter φ is useful to remove noise outside the support of f.

If, for example, ~~during~~ acquisition, noise modifies f on a set of measure zero ~~...~~, the convolution of Rf and φ can be used to remove this effect.

Alternative argument: We claimed in class that if $f = 0$ a.e. in $\mathbb{R}^2$, then $Rf = 0$ a.e.

By contradiction, suppose $\int Rf(s,\omega)\,\varphi(t-s)\,ds \neq 0$ for some $\varphi \in C_c(\mathbb{R})$

then, $\exists$ I interval s.t. $\int_I Rf(s,\omega)\,ds > 0$

Thus $\int_I \int f(t\omega + s\omega^\perp)\,dt\,ds > 0$ $\rightarrow$ CONTRADICTION

3.4.9 Let $\ell \subset \mathbb{R}^2$ be a line segment of length L

We can find a collection of balls of radius $\frac{1}{n}$ s.t. $\ell \subset \bigcup_{i=1}^{N} B_{\frac{1}{n}}(x_i)$, where $N = \frac{nL}{2}$

Here N is obtained by placing equally spaced balls centred on L.

Now: $\sum_{i=1}^{N} \left(\frac{1}{n}\right)^2 = N \frac{1}{n^2} = \frac{nL}{2}\frac{1}{n^2} = \frac{L}{2n} \to 0$ as $n \to \infty$

This shows that the measure of the cover of ℓ can be made arbitrarily small. Hence ℓ has measure 0, according to Def 3.4.5 in book

3.4.11 Let $E = \{(0,\omega) : \omega \in S^1\} \subset \mathbb{R}\times S^1$

Under the map $(t,\theta) \mapsto (t, \omega(\theta))$, the set E is isomorphic to the line segment $\{0\}\times[0,2\pi)$ in $\mathbb{R}\times[0,2\pi)$.

By Ex. 3.4.9, the segment has measure 0 in $\mathbb{R}\times[0,2\pi]$

4.2.7 By direct calculation (in class), when $k=0$, $h_0(x) = e^{-\frac{x^2}{2}}$ and

$$\mathcal{F}(h_0) = \sqrt{2\pi}\, h_0$$

claim: $\lambda_k = (-i)^k \sqrt{2\pi}$

Suppose $\mathcal{F}(h_k) = \lambda_k h_k$

Use the notation $Df(x) = \frac{df}{dx}(x)$, $Mf(x) = x f(x)$. Then: $h_k(x) = (D-M)^k h_0(x)$

We have:

$$\begin{aligned}
\mathcal{F}(h_{k+1}) = \mathcal{F}((D-M)h_k) &= iM\mathcal{F}(h_k) - iD\mathcal{F}(h_k) \\
&= (-i)(D-M)\mathcal{F}(h_k) \\
&= (-i)(D-M)\lambda_k h_k = (-i)(-i)^k\sqrt{2\pi}\,(D-M)h_k \\
&= (-i)^{k+1}\sqrt{2\pi}\, h_{k+1}
\end{aligned}$$

In first line, I used property that $\mathcal{F}(Df) = iM\mathcal{F}(f)$, $\mathcal{F}(Mf) = -iD\mathcal{F}(f)$

These formulae were proven in class.

Note:

$$h_1(x) = (D-M)h_0(x) = -2x\, e^{-x^2/2}$$

$$h_2(x) = (D-M)h_1(x) = (-2+4x^2)\, e^{-x^2/2}$$

$$h_3(x) = (D-M)h_2(x) = 8x\, e^{-x^2/2} - x(-2+4x^2)\, e^{-x^2/2} + (2x - 4x^3)\, e^{-x^2/2} = (12x - 8x^3)\, e^{-x^2/2}$$