HW3 SOLUTION

4.2.14 We will show that $\int_{\mathbb{R}} |\hat{\varphi}(\xi)|(1+|\xi|)^j \, d\xi < \infty$ for all j; (1)

In fact: $$\int_{\mathbb{R}} |\hat{\varphi}(\xi)|(1+|\xi|)^j \, d\xi = \int_{-1}^{1} |\hat{\varphi}(\xi)|(1+|\xi|)^j \, d\xi \leq 2^j \int_{-1}^{1} |\hat{\varphi}(\xi)|^2 \, d\xi < \infty$$

The last inequality follows from the fact that $\hat{\varphi}$ is bounded.

(1) together with the observation that $\varphi \in L^1$ (since φ is rapidly decreasing) imply by Prop. 4.2.2 that $\varphi \in C^j$ for any j. Thus $\varphi \in C^\infty$.

4.2.22 If $f = 0$, then clearly $\langle f, g\rangle = 0$ for all $g \in L^2(\mathbb{R})$.

If $\langle f, g\rangle = 0$ for all $g \in L^2(\mathbb{R})$, then choose $g = f$.

This gives $\langle f, f\rangle = \int_{\mathbb{R}} |f|^2 = 0$ and this implies that $f = 0$ a.e.

This shows that ~~$\langle f,g\rangle=0$~~ $f \in L^2(\mathbb{R})$ is 0 iff $\langle f, g\rangle = 0$ $\forall g \in L^2(\mathbb{R})$

claim $\mathcal{F}(L^2(\mathbb{R})) = L^2(\mathbb{R})$

If $f \in L^2(\mathbb{R})$, then $\|\hat{f}\|_{L^2}^2 = 2\pi \|f\|_{L^2}^2$, hence $\hat{f} \in L^2$

This shows that $\mathcal{F}(L^2(\mathbb{R})) \subset L^2(\mathbb{R})$. Write $\mathcal{F}(L^2(\mathbb{R})) = V \subset L^2(\mathbb{R})$

~~Suppose that there is $g \in L^2(\mathbb{R})$, $g \neq 0$, s.t. $g \notin \mathcal{F}(L^2(\mathbb{R}))$. This means that $g \in \mathcal{F}(L^2(\mathbb{R}))^\perp$ as $\mathcal{F}(L^2(\mathbb{R}))$ is a closed subspace of a Hilbert space~~

~~Thus $\langle g, \hat{f}\rangle = 0$ for all $f \in L^2$~~

This implies $\langle \hat{g}, f\rangle = 0$ ~~for all $f \in L^2$~~

V is a closed subspace. Take $g \in L^2$, $g \notin \mathcal{F}(L^2(\mathbb{R}))$, $g \neq 0$. Hence $g \in V^\perp$

then $\langle g, \hat{f}\rangle = 0$ $\forall f \in L^2(\mathbb{R}^2)$

then $\langle \hat{g}, f\rangle = 0$ $\forall f \in L^2(\mathbb{R}^2)$

Since $\hat{g} \in L^2(\mathbb{R})$, the first part of exercise shows that it must be $\hat{g} = 0 \Rightarrow g = 0$

This is a contradiction. It must be $V^\perp = \{0\}$ or $\mathcal{F}(L^2(\mathbb{R})) = L^2(\mathbb{R})$

4.2.24 Suppose f is odd

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx = \int_{\mathbb{R}} f(x) \cos x\xi \, dx - i \int_{\mathbb{R}} f(x) \sin x\xi \, dx$$

Since $f(x)$ is odd, $f(x)\cos(x\xi)$ is odd and $\int f(x) \cos x\xi \, dx = 0$

Since $f(x)$ is odd, $f(x)\sin(x\xi)$ is even, and $\int_{\mathbb{R}} f(x) \sin x\xi \, dx = 2\int_0^\infty f(x) \sin x\xi \, dx$

thus: $$\hat{f}(\xi) = -2i \int_0^\infty f(x) \sin(x\xi) \, dx$$

4.3.11 By definition, if $f \in L^2$ has a weak L^2-derivative, there exists $f_1 \in L^1_{loc}$ s.t.

$$\int f(x)\,\overline{g'(x)}\,dx = -\int f_1(x)\,g(x)\,dx \qquad \forall\, g \in C^1_c(\mathbb{R})$$

Since f_1 is also in L^2, then we have

$$\int \hat{f}(\xi)\,\overline{(i\xi\,\hat{g}(\xi))}\,d\xi = -\int \hat{f}_1(\xi)\,\overline{\hat{g}(\xi)}\,d\xi \qquad \forall\, g \in C^1_c(\mathbb{R})$$

Thus:

$$\int_{\mathbb{R}} \left(i\xi\,\hat{f}(\xi) - \hat{f}_1(\xi)\right)\overline{\hat{g}(\xi)}\,d\xi = 0 \qquad \forall\, g \in C^1_c(\mathbb{R})$$

NOTE: $C^1_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Thus $\mathcal{F}(C^1_c(\mathbb{R}))$ is dense in $L^2(\mathbb{R})$

If $f \in L^2$ and $\langle f, g\rangle = 0$ $\forall g$ in a dense subspace of L^2, then $f = 0$ a.e.

This implies that $\hat{f}_1(\xi) = i\xi\,\hat{f}(\xi)$ a.e.

[RK Since $\mathcal{F}(C^1_c)$ is dense in L^2, we can choose $\hat{g}(\xi)$ to be a positive bump function with support on a small interval. This can be used to show that $i\xi\hat{f}(\xi) - \hat{f}_1(\xi)$ must be 0 a.e.]

4.5.5 Let $f \in L^1(\mathbb{R})$ $\operatorname{supp} f \subset B_R$

$$\left|\int_{\mathbb{R}^n} (1+\|x\|)^k |f(x)|\,dx\right| = \left|\int_{B_R} (1+\|x\|)^k |f(x)|\,dx\right| \le (1+R)^k \|f\|_{L^1(\mathbb{R}^n)} < \infty$$

By Prop. 4.5.5 this implies that $\hat{f} \in C^k(\mathbb{R}^n)$.

Since k is arbitrary, then $\hat{f} \in C^\infty(\mathbb{R}^n)$.

5.1.9 Suppose there exists $i \in L^1(\mathbb{R})$ s.t.

$$i * f = f \qquad \forall f \in L^1(\mathbb{R})$$

Then $(i*f)^\wedge = \hat{i}\,\hat{f} = \hat{f}$ $\quad \forall f \in L^1(\mathbb{R})$

This implies that $\hat{i} = 1$ a.e.

By Riemann-Lebesgue lemma $\lim_{|\xi|\to\infty} \hat{i}(\xi) = 0$ since $i \in L^1(\mathbb{R})$

This is a contradiction. Hence there was no such funct. $i \in L^1(\mathbb{R})$.