5.1.11 Prove $\widehat{(fg)}(\xi) = \frac{1}{2\pi} \hat{f} * \hat{g}$

By Fourier Inversion Thm., if $f, g \in L^1$, $\hat{f} \in L^1$ (so that f is bounded)

we have:

$$\widehat{fg}(\xi) = \frac{1}{2\pi} \int f(x)\, g(x)\, e^{-ix\xi}\, dx$$

$$= \frac{1}{2\pi} \int g(x)\, e^{-ix\xi}\, dx \int \hat{f}(w)\, e^{ixw}\, dw$$

Since $\hat{f} g \in L^1(\mathbb{R}^2)$

$$\widehat{fg}(\xi) = \frac{1}{2\pi} \int \hat{f}(w)\, dw \int g(x)\, e^{-ix(\xi - w)}\, dx$$

$$= \frac{1}{2\pi} \int \hat{f}(w)\, \hat{g}(\xi - w)\, dw$$

$$= \frac{1}{2\pi} (\hat{f} * \hat{g})(w)$$

We have used of assumptions $f, g \in L^1$, $\hat{f} \in L^1$

6.1.2 Want to show

$$R(\partial_x^i \partial_y^j f)(t, w) = w_1^i w_2^j \partial_t^{i+j} Rf(t, w) \qquad i+j \leq k$$

Case $k=1$ is given by lemma 6.1.1

Assume the statement holds for $k-1$.

Let us now derive the case k. Let $i+j = k$, and assume $j \neq 0$.

$$R(\partial_x^i \partial_y^j f)(t, w) = R(\partial_y \underbrace{\partial_x^i \partial_y^{j-1} f}_{\text{It satisfies hyp of lemma 6.1.1}})(t, w)$$

by lemma 6.1.1 $= w_2 \partial_t R(\partial_x^i \partial_y^{j-1} f)(t, w)$

by INDUCTIVE STEP ($i+j \leq k-1$) $= w_2 w_1^i w_2^{j-1} \partial_t^{i+j} Rf(t, w)$

$= w_1^i w_2^j \partial_t^{i+j} Rf(t, w)$

If $j = 0$, we will assume $i \neq 0$ and we can derive the same conclusion using very similar calculations

Thus the result is proved by INDUCTION.

6.1.6 $\quad Rf(t,\omega) = \int_{\ell_{t,\omega}} f(x)\,ds = \iint_{\mathbb{R}^2} f(x)\,\delta(t-\langle x,\omega\rangle)\,dx$

Thus: $\quad Rf_A(t,\omega) = \iint_{\mathbb{R}^2} f(Ax)\,\delta(t-\langle x,\omega\rangle)\,dx$

Since A is invertible, $\quad y = Ax \Rightarrow x = A^{-1}y \qquad dx = |\det A^{-1}|\,dy$

$$Rf_A(t,\omega) = \cancel{\iint_{\mathbb{R}^2}}\ |\det A^{-1}| \iint_{\mathbb{R}^2} f(y)\,\delta(t-\langle A^{-1}y,\omega\rangle)\,dy$$

$$= |\det A^{-1}| \iint_{\mathbb{R}^2} f(y)\,\delta(t-\langle y,(A^{-1})^T\omega\rangle)\,dy$$

$$= |\det A^{-1}| \iint_{\mathbb{R}^2} f(y)\,\delta\left(\frac{t}{\|(A^{-1})^T\omega\|} - \left\langle y, \frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|}\right\rangle\right)\,dy$$

$$= |\det A^{-1}|\ Rf\left(\frac{t}{\|(A^{-1})^T\omega\|}, \frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|}\right)$$

Alternative argument

$$Rf_A(t,\omega) = \int_{\mathbb{R}} f(tA\omega + sA\omega^\perp)\,ds = \int_{\mathbb{R}} f\left(tA\omega + \tilde{s}\,\frac{A\omega^\perp}{\|A\omega^\perp\|}\right)\frac{d\tilde{s}}{\|A\omega^\perp\|}$$

where $\quad \tilde{s} = \|A\omega^\perp\|\,s$

NOTE: $\quad \langle A\omega^\perp, (A^{-1})^T\omega\rangle = \langle \omega^\perp,\omega\rangle = 0$

WANT TO write $\quad A\omega = c_1\,\frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|} + c_2\,\frac{A\omega^\perp}{\|A\omega^\perp\|}$

We find $\quad c_1 = \left\langle A\omega, \frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|}\right\rangle = \frac{1}{\|(A^{-1})^T\omega\|}$

$$c_2 = \left\langle A\omega, \frac{A\omega^\perp}{\|A\omega^\perp\|}\right\rangle = \frac{1}{\|A\omega^\perp\|}\langle A\omega, A\omega^\perp\rangle$$

Thus

$$Rf_A(t,\omega) = \int_{\mathbb{R}} f\left(t\,\frac{1}{\|(A^{-1})^T\omega\|}\,\frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|} + \left(\frac{\langle A\omega, A\omega^\perp\rangle}{\|A\omega^\perp\|} + \tilde{s}\right)\frac{A\omega^\perp}{\|A\omega^\perp\|}\right)\frac{d\tilde{s}}{\|A\omega^\perp\|}$$

$$= \frac{1}{\|A\omega^\perp\|}\int_{\mathbb{R}} f\left(\frac{t}{\|(A^{-1})^T\omega\|}\,\tilde{\omega} + s'\,\tilde{\omega}^\perp\right)ds'$$

$$\tilde{\omega} = \frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|}, \qquad \tilde{\omega}^\perp = \frac{A\omega^\perp}{\|A\omega^\perp\|}$$

$$\boxed{= \frac{1}{\|A\omega^\perp\|}\ Rf\left(\frac{t}{(A^{-1})^T\omega}, \frac{(A^{-1})^T\omega}{\|(A^{-1})^T\omega\|}\right)}$$

6.2.3 Let f be differentiable with bounded support

$$\partial_t (\mathcal{H}f)(t) = \frac{\partial}{\partial t}\,\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(r)\,\mathrm{sgn}(r)\,e^{itr}\,dr$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(r)\,\mathrm{sgn}(r)\,(ir)\,e^{itr}\,dr$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(ir\,\hat{f}(r)\right)(\mathrm{sgn}\, r)\,e^{itr}\,dr$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} (\partial_t f)^{\wedge}(r)\,(\mathrm{sgn}\, r)\,e^{itr}\,dr$$

$$= \mathcal{H}(\partial_t f)(t)$$

6.2.5 Since f has bounded support, by Prop. 4.2.3, $\hat{f}$ must be a C^∞ function, with derivatives tending to 0 as $|\xi| \to \infty$.

If $\hat{f}(0) \neq 0$, then, in a neighborhood of the origin $\hat{f}(\xi)$ must be either always ~~either~~ positive or always negative.

Suppose $\mathcal{H}f$ has bounded support. Then, by the observation above, $(\mathcal{H}f)^{\wedge}(\xi)$ must be either always positive or always negative in a neighborhood of 0. But this is impossible since $(\mathcal{H}f)^{\wedge}(\xi) = \hat{f}(\xi)\,\mathrm{sgn}(\xi)$.

Thus $\mathcal{H}f$ must have unbounded support.

~~By the same argument, $\mathcal{H}f$~~ Since $(\mathcal{H}f)^{\wedge}$ is discontinuous at the origin, it cannot have rapid decay. In fact $\mathcal{H}f$ is NOT even in L^1

[If $\mathcal{H}f \in L^1$, then $(\mathcal{H}f)^{\wedge}$ would be continuous] $\hookrightarrow$ $\mathcal{H}f$ decays at most as $O(x^{-1})$ as $|x| \to \infty$.

6.2.8 Let φ be a C^1 test function. Let $F(t) = \frac{1}{i}\mathcal{H}\,\mathcal{R}f(t,\omega) = (6.32)$

$$\int F(t)\,\varphi'(t)\,dt = \int_{-\infty}^{-1} 2\left(t+\sqrt{t^2-1}\right)\varphi'(t)\,dt + \int_{-1}^{1} 2t\,\varphi'(t)\,dt + \int_{1}^{\infty} 2\left(t-\sqrt{t^2-1}\right)\varphi'(t)\,dt$$

$\boxed{\int F\varphi' = \int -F_1\varphi}$

$F_1 \leftarrow$ weak derivative

$$= \int_{-\infty}^{\infty} 2t\,\varphi'(t) + \int_{-\infty}^{-1} 2\sqrt{t^2-1}\,\varphi'(t)\,dt - \int_{1}^{\infty} 2\sqrt{t^2-1}\,\varphi'(t)\,dt =$$

$$= \cancel{2t\,\varphi(t)\Big|_{-\infty}^{\infty}} - 2\int_{-\infty}^{\infty}\varphi(t)\,dt + \cancel{2\sqrt{t^2-1}\,\varphi(t)\Big|_{-\infty}^{-1}} - \int_{-\infty}^{-1}\frac{2t}{\sqrt{t^2-1}}\,\varphi(t)\,dt +$$

$$\cancel{-2\sqrt{t^2-1}\,\varphi(t)\Big|_{1}^{\infty}} + \int_{1}^{\infty}\frac{2t}{\sqrt{t^2-1}}\,\varphi(t)\,dt =$$

$$= -2\int_{-\infty}^{\infty}\varphi(t)\,dt + \int_{-\infty}^{-1}\frac{2|t|}{\sqrt{t^2-1}}\,dt + \int_{1}^{\infty}\frac{2|t|}{\sqrt{t^2-1}}\,dt \implies F_1(t) = \begin{cases} 2 & |t|<1 \\ 2-\dfrac{2|t|}{\sqrt{t^2-1}} & |t|\geq 1 \end{cases}$$

WEAK DERIVATIVE