Math 6397 – Spring 2015 Name:

TEST #1

Closed-book test. Please, write legibly.

(1)[2 Pts] State the definition of the Radon transform of f on $\mathbb{R}^2$. Illustrate the variables involved in the definition with a sketch.

(2)[4 Pts] Consider the function

$$f(x,y) = \begin{cases} 1 & \text{if } 4x^2 + y^2 \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

that is, f has the value 1 inside the ellipse $4x^2 + y^2 = 4$

(a) Compute the Radon transform for $\theta = 0$, that is, $Rf(t, \theta = 0)$

(b) Sketch the support of the Radon transform of f in $[0, \pi] \times \mathbb{R}$.

(c) [Extra credit 2 Pts] Derive an analytic expression for the boundary curve of the support region.

(3)[1 Pts] State the definition of the Fourier transform of f on $\mathbb{R}^2$.

(4)[3 Pts] For each of the following statements, say if it is true or false. If true, briefly say why; if not give a counterexample.

(a) If $f \in L^1(\mathbb{R})$, then $\hat{f} \in L^1(\mathbb{R})$.

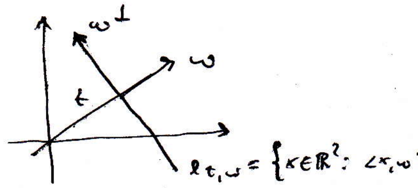
(b) If $f \in L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$.

(c) If $f \in L^1(\mathbb{R})$ is real-valued, then $\hat{f}$ is real-valued.

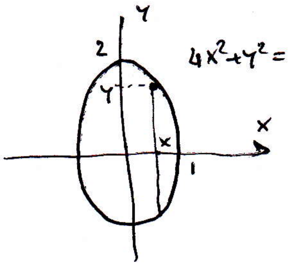
TEST #1 SOLUTION

(1) For $f \in L^1(\mathbb{R}^2)$, the RADON TRANSFORM of f at $(t,\omega) \in \mathbb{R} \times S^1$ is

$$Rf(t,\omega) = \int_{\ell_{t,\omega}} f\, ds = \int_{\mathbb{R}} f(t\omega + s\omega^\perp)\, ds$$



(2)



$Rf(t,\omega(\theta)) = \ell_{t,\omega} \cap E$ where $E = \{(x,y) \in \mathbb{R}^2 : 4x^2+y^2 \le 4\}$

(a) For $\theta = 0$, $Rf(t,0)$ is the (vertical) width of the ellipse, i.e., twice the positive y coordinate corresponding to x.

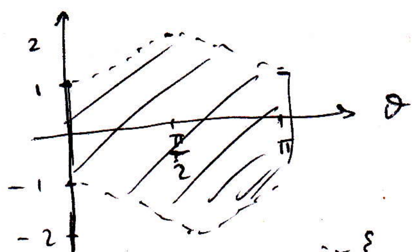
$$y^2 = 4 - 4x^2 \Rightarrow y = 2\sqrt{1-x^2}$$

$$Rf(t,\omega(0)) = \begin{cases} 4\sqrt{1-t^2} & |t| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $\theta = 0$, $Rf(t,\omega(0)) = 0$ if $|t| > 1$. Same for $\theta = \pi$

For $\theta = \frac{\pi}{2}$ $Rf(t,\omega(\frac{\pi}{2})) = 0$ if $|t| > 2$

For $0 < \theta < \frac{\pi}{2}$, the t-support of $Rf(t,\omega(\theta))$ is between 1 and 2



(3) For $f \in L^1(\mathbb{R})$, $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$

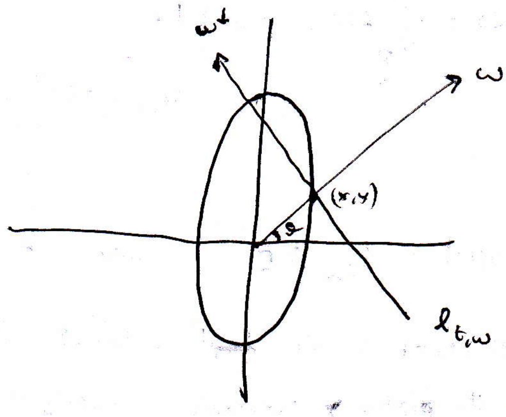
For $f \in L^1(\mathbb{R}^2)$, $\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-i\langle x,\xi\rangle}\, dx$

(4) (a) FALSE. If $f = \chi_{[-1,1]}$, then $\hat{f}(\xi) = 2\,\mathrm{sinc}(\xi) \notin L^1(\mathbb{R})$

(b) TRUE. By Parseval Formula

(c) FALSE. f need to be real + even. If $f = \chi_{[0,2]}$, then $\hat{f}(\xi) = 2\,\mathrm{sinc}(\xi) e^{-i\xi}$

2c



$\ell_{t,w^\perp} = \{(x, x\tan\theta)\}$

The intersectn of ellipse and $\ell_{t,w^\perp}$ is

$$\begin{cases} 4x^2+y^2=4 \\ y = x\tan\theta \end{cases} \Rightarrow \quad x^2 = \frac{4}{4+\tan^2\theta}$$

$$y^2 = \frac{4\tan^2\theta}{4+\tan^2\theta}$$

$$t^2 = x^2+y^2 = \frac{4(1+\tan^2\theta)}{4+\tan^2\theta} = \frac{4}{4\cos^2\theta+\sin^2\theta}$$

thus

$$t = \pm\frac{2}{\sqrt{4\cos^2\theta+\sin^2\theta}}$$

Note:

$\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{2} \Rightarrow t = 2$