2. Mark each statement True or False. Justify each answer.

(a) If a convergent sequence is bounded, then it is monotone.
(b) If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.
(c) The Cauchy convergence criterion holds in $\mathbb{Q}$, the ordered field of rational numbers.

3. Prove that each sequence is monotone and bounded. Then find the limit. ☆

(a) $s_1 = 1$ and $s_{n+1} = \frac{1}{5}(s_n + 7)$ for $n \ge 1$.

(b) $s_1 = 2$ and $s_{n+1} = \frac{1}{5}(s_n + 7)$ for $n \ge 1$.

(c) $s_1 = 2$ and $s_{n+1} = \frac{1}{4}(2s_n + 7)$ for $n \ge 1$.

(d) $s_1 = 1$ and $s_{n+1} = \sqrt{2s_n + 2}$ for $n \ge 1$.

(e) $s_1 = 5$ and $s_{n+1} = \sqrt{4s_n + 1}$ for $n \ge 1$.

4. Find an example of a sequence of real numbers satisfying each set of properties.

(a) Cauchy, but not monotone
(b) Monotone, but not Cauchy
(c) Bounded, but not Cauchy

5. Let (a_n) and (b_n) be monotone sequences. Prove or give a counterexample.

(a) The sequence (c_n) given by $c_n = a_n + b_n$ is monotone.
(b) The sequence (c_n) given by $c_n = a_n \cdot b_n$ is monotone.

6. Let (a_n) and (b_n) be monotone sequences. Prove or give a counterexample.

(a) The sequence (c_n) given by $c_n = ka_n$ is monotone for any $k \in \mathbb{R}$.
(b) The sequence (c_n) given by $c_n = a_n / b_n$ is monotone, where $b_n \neq 0$ for all $n \in \mathbb{N}$.

7. Let $s_1 = \sqrt{6}$, $s_2 = \sqrt{6+\sqrt{6}}$, $s_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$, and in general define $s_{n+1} = \sqrt{6+s_n}$. Prove that (s_n) converges, and find its limit.

8. Let $s_1 = k$ and define $s_{n+1} = \sqrt{4s_n - 1}$ for $n \ge 1$. Determine for what values of k the sequence (s_n) will be monotone increasing and for what values of k it will be monotone decreasing.

9. Suppose $x > 0$. Define a sequence (s_n) by $s_1 = k$ and $s_{n+1} = (s_n^2 + x)/(2s_n)$ for $n \in \mathbb{N}$. Prove that for any $k > 0$, $\lim s_n = \sqrt{x}$. ☆

10. (a) Suppose that $|r| < 1$. Also, we know that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Find $\lim_{n\to\infty} (1 + r + r^2 + \cdots + r^n)$.

(b) If we let the infinite repeating decimal $0.9999\cdots$ stand for the limit

4. For each sequence, find the set S of subsequential limits, the limit superior, and the limit inferior.

(a) $w_n = \dfrac{(-1)^n}{n}$

(b) $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \ldots)$

(c) $y_n = n[2 + (-1)^n]$

(d) $z_n = (-n)^n$

5. Use Exercise 3.14 to find the limit of each sequence. ☆

(a) $s_n = \left(1+\dfrac{1}{2n}\right)^{2n}$ (b) $s_n = \left(1+\dfrac{1}{n}\right)^{2n}$

(c) $s_n = \left(1+\dfrac{1}{n}\right)^{n-1}$ (d) $s_n = \left(\dfrac{n}{n+1}\right)^{n}$

(e) $s_n = \left(1+\dfrac{1}{2n}\right)^{n}$ (f) $s_n = \left(\dfrac{n+2}{n+1}\right)^{n+3}$

6. Prove or give a counterexample.

(a) Every oscillating sequence has a convergent subsequence.
(b) Every oscillating sequence diverges.
(c) Every divergent sequence oscillates.

7. Prove or give a counterexample.

(a) Every bounded sequence has a Cauchy subsequence.
(b) Every monotone sequence has a bounded subsequence.
(c) Every convergent sequence can be represented as the sum of two oscillating sequences.

8. If (s_n) is a subsequence of (t_n) and (t_n) is a subsequence of (s_n), can we conclude that $(s_n) = (t_n)$? Prove or give a counterexample.

9. Let (s_n) be a bounded sequence and suppose that $\liminf s_n = \limsup s_n = s$. Prove that (s_n) is convergent and that $\lim s_n = s$. ☆

***10.** Suppose that $x > 1$. Prove that $\lim x^{1/n} = 1$.

11. Let (s_n) be a bounded sequence and let S denote the set of subsequential limits of (s_n). Prove that S is closed. ☆

12. Let $A = \{x \in \mathbb{Q}: 0 \le x < 2\}$. Since A is denumerable, there exists a bijection $s: \mathbb{N} \to A$. Letting $s(n) = s_n$, find the set of subsequential limits of the sequence (s_n).

13. Let (s_n) and (t_n) be bounded sequences.

(a) Prove that $\limsup (s_n + t_n) \le \limsup s_n + \limsup t_n$. ☆