4. For each sequence, find the set S of subsequential limits, the limit superior, and the limit inferior.

(a) $w_n = \dfrac{(-1)^n}{n}$

(b) $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \ldots)$

(c) $y_n = n[2 + (-1)^n]$

(d) $z_n = (-n)^n$

5. Use Exercise 3.14 to find the limit of each sequence. ☆

(a) $s_n = \left(1+\dfrac{1}{2n}\right)^{2n}$ (b) $s_n = \left(1+\dfrac{1}{n}\right)^{2n}$

(c) $s_n = \left(1+\dfrac{1}{n}\right)^{n-1}$ (d) $s_n = \left(\dfrac{n}{n+1}\right)^{n}$

(e) $s_n = \left(1+\dfrac{1}{2n}\right)^{n}$ (f) $s_n = \left(\dfrac{n+2}{n+1}\right)^{n+3}$

6. Prove or give a counterexample.

(a) Every oscillating sequence has a convergent subsequence.
(b) Every oscillating sequence diverges.
(c) Every divergent sequence oscillates.

7. Prove or give a counterexample.

(a) Every bounded sequence has a Cauchy subsequence.
(b) Every monotone sequence has a bounded subsequence.
(c) Every convergent sequence can be represented as the sum of two oscillating sequences.

8. If (s_n) is a subsequence of (t_n) and (t_n) is a subsequence of (s_n), can we conclude that $(s_n) = (t_n)$? Prove or give a counterexample.

9. Let (s_n) be a bounded sequence and suppose that $\liminf s_n = \limsup s_n = s$. Prove that (s_n) is convergent and that $\lim s_n = s$. ☆

***10.** Suppose that $x > 1$. Prove that $\lim x^{1/n} = 1$.

11. Let (s_n) be a bounded sequence and let S denote the set of subsequential limits of (s_n). Prove that S is closed. ☆

12. Let $A = \{x \in \mathbb{Q}: 0 \le x < 2\}$. Since A is denumerable, there exists a bijection $s: \mathbb{N} \to A$. Letting $s(n) = s_n$, find the set of subsequential limits of the sequence (s_n).

13. Let (s_n) and (t_n) be bounded sequences.

(a) Prove that $\limsup (s_n + t_n) \le \limsup s_n + \limsup t_n$. ☆

(b) Find an example to show that equality may not hold in part (a).

14. State and prove the analog of Theorem 4.11 for lim inf.

15. Let (s_n) and (t_n) be bounded sequences.
(a) Prove that $\liminf s_n + \liminf t_n \leq \liminf (s_n + t_n)$.
(b) Find an example to show that equality may not hold in part (a).

16. Let (s_n) be a bounded sequence.
(a) Prove that $\limsup s_n = \lim_{N\to\infty} \sup \{s_n: n > N\}$.
(b) Prove that $\liminf s_n = \lim_{N\to\infty} \inf \{s_n: n > N\}$.

***17.** Prove that if $\limsup s_n = +\infty$ and $k > 0$, then $\limsup (ks_n) = +\infty$.

18. Let C be a nonempty subset of $\mathbb{R}$. Prove that C is compact iff every sequence in C has a subsequence that converges to a point in C.

19. Prove that every sequence has a monotone subsequence.

Hints for Selected Exercises

Section 1

3. (a) 1, 4, 9, 16, 25, 36, 49; (c) $\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}$.

7. (d) $\sqrt{n}/(n+1) < \sqrt{n}/n = 1/\sqrt{n}$.
(f) Suppose first that $x \neq 0$. Since $|x| < 1$, $|x| = 1/(1+y)$ for some $y > 0$. Use Bernoulii's inequality to obtain the bound $|x^n| = 1/(1+y)^n < 1/(ny)$. The case when $x = 0$ is trivial.

9. (a) True
(b) False; (c) True.

13. Use the fact that $a_n \to b$ and $c_n \to b$ to show that given any $\varepsilon > 0$, there exists a natural number N such that $b - \varepsilon < a_n \leq b_n \leq c_n < b + \varepsilon$, whenever $n \geq N$. Note that this N must work for *both* sequences (a_n) and (c_n).

15. (a) Suppose that (s_n) is a sequence in $S\setminus\{x\}$ such that $s_n \to x$. Given any deleted neighborhood $N^*(x;\varepsilon)$, find an integer k such that $s_k \in N^*(x;\varepsilon)$.

For the converse, suppose that x is an accumulation point of S. Given any $\varepsilon > 0$, there exists a point in $S \cap N^*(x;\varepsilon)$. Thus for each $n \in \mathbb{N}$ there exists a point, say s_n, in $S \cap N^*(x;1/n)$. Clearly, $s_n \in S\setminus\{x\}$ for all n. Now show that $s_n \to x$. Note that this half of the argument uses the axiom of choice.

17. Let $t_n = (s_n - s)/(s_n + s)$ and solve for s_n.

(b) Suppose that $m \in \mathbb{N}$ with $m > 1$ and that $\sum_{n=m}^{\infty} a_n$ is convergent. If $a_1, \ldots, a_{m-1}$ are real numbers, prove that $\sum_{n=1}^{\infty} a_n$ is convergent and that $\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{m-1} + \sum_{n=m}^{\infty} a_n$.

4. Show that each series is divergent.

(a) $\sum (-1)^n$ (b) $\sum \dfrac{n}{2n+1}$

(c) $\sum \dfrac{n}{\sqrt{n^2+1}}$ (d) $\sum \cos \dfrac{n\pi}{2}$

5. Find the sum of each series. ☆

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ (b) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$ (d) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n$

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ (j) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(n+1)}$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (l) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+k)!}$ where $k \in \mathbb{N}$ is fixed.

6. Prove Theorem 1.4.

***7.** Given the series $\sum a_n$ and $\sum b_n$, suppose that there exists a natural number N such that $a_n = b_n$ for all $n \geq N$. Prove that $\sum a_n$ is convergent iff $\sum b_n$ is convergent. Thus the convergence of a series is not affected by changing a finite number of terms. (Of course, the value of the sum may change.) ☆

***8.** Let (a_n) be a sequence of nonnegative real numbers. Prove that $\sum a_n$ converges iff the sequence of partial sums is bounded.

9. Determine whether or not the series $\sum_{n=1}^{\infty} 1/(\sqrt{n+1}+\sqrt{n})$ converges. Justify your answer. ☆

10. Let (x_n) be a sequence of real numbers and let $y_n = x_n - x_{n+1}$ for each $n \in \mathbb{N}$.

(a) Prove that the series $\sum_{n=1}^{\infty} y_n$ converges iff the sequence (x_n) converges.

(b) If $\sum_{n=1}^{\infty} y_n$ converges, what is the sum?