6. Use Definition 1.1 to prove each limit.

(a) $\lim_{x\to 3} (x^2 - 5x + 1) = -5$

(b) $\lim_{x\to -3} (x^2 + 3x + 8) = 8$

(c) $\lim_{x\to 2} x^3 = 8$

7. Find the following limits and prove your answers.

(a) $\lim\limits_{x\to 0} |x|$

(b) $\lim\limits_{x\to 0} x^2/|x|$

(c) $\lim\limits_{x\to c} \sqrt{x}$, where $c \geq 0$. ✰

8. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that $\lim_{x\to c} f(x) = L$.

(a) Prove that $\lim_{x\to c} |f(x)| = |L|$.

(b) If $f(x) \geq 0$ for all $x \in D$, prove that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

9. Determine whether or not the following limits exist. Justify your answers. ✰

(a) $\lim\limits_{x\to 0+} \dfrac{1}{x}$

(b) $\lim\limits_{x\to 0+} \left|\sin \dfrac{1}{x}\right|$

(c) $\lim\limits_{x\to 0+} x \sin \dfrac{1}{x}$

10. Prove Corollary 1.9

(a) by using Definition 1.1.

(b) by using Theorem 1.8 and the "Limit of a Sequence" theorem "If a sequence converges, its limit is unique.".

11. Prove Theorem 1.10. ✰

12. Finish the proof of Theorem 1.13.

13. Let f, g, and h be functions from D into $\mathbb{R}$, and let c be an accumulation point of D. Suppose that $f(x) \leq g(x) \leq h(x)$, for all $x \in D$ with $x \neq c$, and suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$. Prove that $\lim_{x\to c} g(x) = L$. ✰

14. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that $a \leq f(x) \leq b$ for all $x \in D$ with $x \neq c$, and suppose that $\lim_{x\to c} f(x) = L$. Prove that $a \leq L \leq b$.

15. Let f and g be functions from D into $\mathbb{R}$ and let c be an accumulation point of D. Suppose that there exist a neighborhood U of c and a real number M such that $|g(x)| \leq M$ for all $x \in U \cap D$. If $\lim_{x\to c} f(x) = 0$, prove that $\lim_{x\to c} (fg)(x) = 0$. ✰

(b) If $f(D)$ is a bounded set, then f is continuous on D.
(c) If c is an isolated point of D, then f is continuous at c.
(d) If f is continuous at c and (x_n) is a sequence in D, then $x_n \to c$ whenever $f(x_n) \to f(c)$.
(e) If f is continuous at c, then for every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) = V$.

2. Let $f: D \to \mathbb{R}$ and let $c \in D$. Mark each statement True or False. Justify each answer.

(a) If f is continuous at c and c is an accumulation point of D, then $\lim_{x \to c} f(x) = f(c)$.
(b) Every polynomial is continuous at each point in $\mathbb{R}$.
(c) If (x_n) is a Cauchy sequence in D, then $\big(f(x_n)\big)$ is convergent.
(d) If $f: \mathbb{R} \to \mathbb{R}$ is continuous at each irrational number, then f is continuous on $\mathbb{R}$.
(e) If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are both continuous (on $\mathbb{R}$), then $f \circ g$ and $g \circ f$ are both continuous on $\mathbb{R}$.

3. Let $f(x) = (x^2 + 4x - 21)/(x - 3)$ for $x \neq 3$. How should $f(3)$ be defined so that f will be continuous at 3? ☆

4. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 3x - 5$. Use Definition 2.1 to prove that f is continuous at 3.

5. Find an example of a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at exactly one point. ☆

6. Prove or give a counterexample for each statement.

(a) If f is continuous on D and $k \in \mathbb{R}$, then kf is continuous on D.
(b) If f and $f+g$ are continuous on D, then g is continuous on D.
(c) If f and fg are continuous on D, then g is continuous on D.
(d) If f^2 is continuous on D, then f is continuous on D.
(e) If f is continuous on D and D is bounded, then $f(D)$ is bounded.
(f) If f and g are not continuous on D, then $f+g$ is not continuous on D.
(g) If f and g are not continuous on D, then fg is not continuous on D.
(h) If $f: D \to E$ and $g: E \to F$ are not continuous on D and E, respectively, then $g \circ f: D \to F$ is not continuous on D.

7. Prove or give a counterexample: Every sequence of real numbers is a continuous function. ☆

8. Consider the formula

$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n}.$$

Let $D = \{x : f(x) \in \mathbb{R}\}$. Calculate $f(x)$ for all $x \in D$ and determine where $f: D \to \mathbb{R}$ is continuous.

9. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x) = 5x$ if x is rational and $f(x) = x^2 + 6$ if x is irrational. Prove that f is discontinuous at 1 and continuous at 2. Are there any other points besides 2 at which f is continuous? ☆

***10.** (a) Let $f: D\to\mathbb{R}$ and define $|f|: D\to\mathbb{R}$ by $|f|(x) = |f(x)|$. Suppose that f is continuous at $c \in D$. Prove that $|f|$ is continuous at c.
(b) If $|f|$ is continuous at c, does it follow that f is continuous at c? Justify your answer.

***11.** Define max (f, g) and min (f, g) as in Example 2.11. Show that

$\max(f,g) = \frac{1}{2}(f+g)+\frac{1}{2}|f-g|$ and $\min(f,g) = \frac{1}{2}(f+g)-\frac{1}{2}|f-g|$. ☆

12. Let $f: D\to\mathbb{R}$ and suppose that $f(x)\ge 0$ for all $x\in D$. Define $\sqrt{f}: D\to\mathbb{R}$ by $\sqrt{f}(x)=\sqrt{f(x)}$. If f is continuous at $c \in D$, prove that $\sqrt{f}$ is continuous at c.

***13.** Let $f: D\to\mathbb{R}$ be continuous at $c\in D$ and suppose that $f(c) > 0$. Prove that there exists an $\alpha > 0$ and a neighborhood U of c such that $f(x) > \alpha$ for all $x\in U\cap D$. ☆

14. Let $f: D\to\mathbb{R}$ be continuous at $c\in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \le M$ for all $x\in U\cap D$.

15. Complete the proof of Theorem 2.14 by showing that $H\cap D = f^{-1}(G)$.

***16.** Let $f:\mathbb{R}\to\mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ iff $f^{-1}(H)$ is a closed set whenever H is a closed set.

17. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $f(x+y) = f(x)+f(y)$ for all $x, y\in\mathbb{R}$. Prove that there exists $k\in\mathbb{R}$ such that $f(x) = kx$, for every $x\in\mathbb{R}$. ☆

18. Suppose that $f:(a,b)\to\mathbb{R}$ is continuous and that $f(r) = 0$ for every rational number $r\in(a,b)$. Prove that $f(x) = 0$ for all $x\in(a,b)$.

19. Suppose $1\le c\le\sqrt[3]{3}$ and define a sequence (s_n) recursively by $s_1 = c$ and $s_{n+1} = c^{s_n}$ for all $n\in\mathbb{N}$.
(a) Prove that (s_n) is an increasing sequence.
(b) Prove that (s_n) is bounded above.
(c) Prove that (s_n) converges to a number b such that $b = c^b$.
(d) Find the value of the continued power $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$.

3 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the chapter.*

1. Mark each statement True or False. Justify each answer.

(a) Let D be a compact subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.
(b) Suppose that $f: D \to \mathbb{R}$ is continuous. Then, there exists a point x_1 in D such that $f(x_1) \geq f(x)$ for all $x \in D$.
(c) Let D be a bounded subset of $\mathbb{R}$ and suppose that $f: D \to \mathbb{R}$ is continuous. Then $f(D)$ is bounded.

2. Mark each statement True or False. Justify each answer.

(a) Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(c) = 0$.
(b) Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) \leq k \leq f(b)$. Then there exists a point $c \in [a, b]$ such that $f(c) = k$.
(c) If $f: D \to \mathbb{R}$ is continuous and bounded on D, then f assumes maximum and minimum values on D.

3. Let $f: D \to \mathbb{R}$ be continuous. For each of the following, prove or give a counterexample. ☆

(a) If D is open, then $f(D)$ is open.
(b) If D is closed, then $f(D)$ is closed.
(c) If D is not open, then $f(D)$ is not open.
(d) If D is not closed, then $f(D)$ is not closed.
(e) If D is not compact, then $f(D)$ is not compact.
(f) If D is unbounded, then $f(D)$ is unbounded.
(g) If D is finite, then $f(D)$ is finite.
(h) If D is infinite, then $f(D)$ is infinite.
(i) If D is an interval, then $f(D)$ is an interval.
(j) If D is an interval that is not open, then $f(D)$ is an interval that is not open.

4. Show that $3^x = 5x$ for some $x \in (0, 1)$.

5. Show that the equation $5^x = x^4$ has at least one real solution.

6. Show that any polynomial of odd degree has at least one real root.

7. Suppose that $f: [a, b] \to [a, b]$ is continuous. Prove that f has a **fixed point.** That is, prove that there exists $c \in [a, b]$ such that $f(c) = c$. ☆

8. Suppose that $f: [a, b] \to \mathbb{R}$ and $g: [a, b] \to \mathbb{R}$ are continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$.