9. Suppose $f: [a,b] \to \mathbb{R}$ is continuous and that $f([a,b]) \subseteq \mathbb{Q}$. Prove that f is constant on $[a,b]$.

10. Suppose that $f: [a,b] \to \mathbb{R}$ is two-to-one. That is, for each $y \in \mathbb{R}$, $f^{-1}(\{y\})$ either is empty or contains exactly two points.

(a) Find an example of such a function.
(b) Prove that no such function can be continuous.

11. (a) Let $p \in \mathbb{R}$ and define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - p|$. Prove that f is continuous.
(b) Let S be a nonempty compact subset of $\mathbb{R}$ and let $p \in \mathbb{R}$. Prove that S has a "closest" point to p. That is, prove that there exists a point q in S such that $|q - p| = \inf\{|x - p|: x \in S\}$.

12. Prove Theorem 3.2 using the Heine–Borel theorem and the Bolzano–Weierstrass theorem for sequences instead of the open-cover property of compactness.

***13.** Let f be a function defined on an interval I. We say that f is **strictly increasing** if $x_1 < x_2$ in I implies that $f(x_1) < f(x_2)$. Similarly, f is **strictly decreasing** if $x_1 < x_2$ in I implies that $f(x_1) > f(x_2)$. Prove the following.

(a) If f is continuous and injective on I, then f is strictly increasing or strictly decreasing.
(b) If f is strictly increasing and if $f(I)$ is an interval, then f is continuous. Furthermore, f^{-1} is a strictly increasing continuous function on $f(I)$.

14. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) Show that f is not continuous at 0.
(b) Show that f has the intermediate value property on $\mathbb{R}$.

15. Let $f: D \to \mathbb{R}$ and let $c \in D$. We say that f is **bounded on a neighborhood** of c if there exists a neighborhood U of c and a number M such that $|f(x)| \leq M$ for all $x \in U \cap D$.

(a) Suppose that f is bounded on a neighborhood of each x in D and that D is compact. Prove that f is bounded on D. ☆
(b) Suppose that f is bounded on a neighborhood of each x in D, but that D is not compact. Show that f is not necessarily bounded on D, even when f is continuous.
(c) Suppose that $f: [a,b] \to \mathbb{R}$ has a limit at each x in $[a,b]$. Prove that f is bounded on $[a,b]$.

16. A subset S of $\mathbb{R}$ is said to be **disconnected** if there exist disjoint open sets U and V in $\mathbb{R}$ such that $S \subseteq U \cup V$, $S \cap U \neq \varnothing$, and $S \cap V \neq \varnothing$. If S is not disconnected, then it is said to be **connected**. Suppose that S is connected and that $f: \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $f(S)$ is connected. (*Hint*: Use Corollary 2.15.)

4. Prove that each function is uniformly continuous on the given set by directly verifying the ε–δ property in Definition 4.1.

(a) $f(x) = x^3$ on $[0,2]$

(b) $f(x) = \dfrac{1}{x}$ on $[2,\infty)$

(c) $f(x) = \dfrac{x-1}{x+1}$ on $[0,\infty)$

5. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. ☆

6. Let f and g be real-valued functions that are uniformly continuous on D. Prove that $f + g$ is uniformly continuous on D.

7. Let $f: D \to \mathbb{R}$ be uniformly continuous on D and let $k \in \mathbb{R}$. Prove that the function kf is uniformly continuous on D.

8. Let f and g be real-valued functions that are uniformly continuous on D, and suppose that $g(x) \neq 0$ for all $x \in D$.

(a) Find an example to show that the function f/g need not be uniformly continuous on D.

(b) Prove that if D is compact, then f/g must be uniformly continuous on D.

9. Prove or give a counterexample: If $f: A \to B$ is uniformly continuous on A and $g: B \to C$ is uniformly continuous on B, then $g \circ f: A \to C$ is uniformly continuous on A. ☆

10. Find two real-valued functions f and g that are uniformly continuous on a set D, but such that their product fg is not uniformly continuous on D.

11. Let $f: D \to \mathbb{R}$ be uniformly continuous on the bounded set D. Prove that f is bounded on D. ☆

12. (a) Let f and g be real-valued functions that are bounded and uniformly continuous on D. Prove that their product fg is uniformly continuous on D.

(b) Let f and g be real-valued functions that are uniformly continuous on a bounded set D. Prove that their product fg is uniformly continuous on D.

13. Suppose that f is uniformly continuous on $[a,b]$ and uniformly continuous on $[b,c]$. Prove that f is uniformly continuous on $[a,c]$.

14. Prove Theorem 4.6 by justifying the following steps.

(a) Suppose that f is not uniformly continuous on D. Then there exists an $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, there exist x_n and y_n in D with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$.

(b) This part has been intentionally excluded from this edition.

(c) Show that $\lim_{k\to\infty} y_{n_k} = x$.

(d) Show that $(f(x_{n_k}))$ and $(f(y_{n_k}))$ both converge to $f(x)$, to obtain a contradiction.

15. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **periodic** if there exists a number $k > 0$ such that $f(x + k) = f(x)$ for all $x \in \mathbb{R}$. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and periodic. Prove that f is bounded and uniformly continuous on $\mathbb{R}$.

Section 5 CONTINUITY IN METRIC SPACES†

This section discusses the general setting of a metric space.

5.1 DEFINITION A sequence (s_n) in a metric space (X, d) is said to **converge** if there exists a point $s \in X$ such that

> for every $\varepsilon > 0$ there exists a natural number N such that $n \geq N$ implies that $d(s_n, s) < \varepsilon$.

In this case we say that (s_n) converges to s, and we write $s_n \to s$ or $\lim s_n = s$.

For a fixed point s in X, we can think of the real numbers $d(s_n, s)$ as a sequence in $\mathbb{R}$. Thus, to show that a sequence (s_n) converges to s in the metric space (X, d), it suffices to show that the real sequence $(d(s_n, s))$ converges to 0 in $\mathbb{R}$. Furthermore, since $d(s_n, s) \geq 0$ for all n, we can do this by finding a positive real sequence (a_n) such that $d(s_n, s) \leq a_n$ for all n and $a_n \to 0$.

†This section may be skipped, if desired, since it is not used in later sections.