3. Determine if each function is differentiable at $x = 1$. If it is, find the derivative. If not, explain why not.

(a) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

(b) $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

(c) $f(x) = \begin{cases} 3x-2 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

4. Use Definition 1.1 to find the derivative of each function.

(a) $f(x) = 3x + 5$ for $x \in \mathbb{R}$

(b) $f(x) = x^3$ for $x \in \mathbb{R}$

(c) $f(x) = \dfrac{1}{x}$ for $x \neq 0$

(d) $f(x) = \sqrt{x}$ for $x > 0$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$ for $x > 0$

5. Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$.

(a) Use Definition 1.1 to prove that $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$. ☆

(b) Show that f is not differentiable at $x = 0$.

***6.** Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

(a) Use the chain rule and the product rule to show that f is differentiable at each $c \neq 0$ and find $f'(c)$. (You may assume that the derivative of $\sin x$ is $\cos x$ for all $x \in \mathbb{R}$.)

(b) Use Definition 1.1 to show that f is differentiable at $x = 0$ and find $f'(0)$.

(c) Show that f' is not continuous at $x = 0$.

(d) Let $g(x) = x^2$ if $x \leq 0$ and $g(x) = x^2 \sin(1/x)$ if $x > 0$. Determine whether or not g is differentiable at $x = 0$. If it is, find $g'(0)$.

7. Determine for which values of x each function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative.

(a) $f(x) = |x - 1|$

(b) $f(x) = |x^2 - 1|$ ☆

(c) $f(x) = |x|$

(d) $f(x) = x|x|$ ☆

***8.** Let $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that f is differentiable on $\mathbb{R}$.

(b) Show that f' is not bounded on the interval $[-1, 1]$.

9. Let $f(x) = x^2$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$.

(a) Show that f is differentiable at $x = 0$. ☆

(b) Find $f'(x)$ for all real x and sketch the graph of f'.

(c) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$? ☆

10. Complete the proof of parts (a) and (b) of Theorem 1.7.

11. Let $f(x) = x^2$ if x is rational and $f(x) = 0$ if x is irrational.

(a) Prove that f is continuous at exactly one point, namely at $x = 0$.

(b) Prove that f is differentiable at exactly one point, namely at $x = 0$.

12. Prove: If a polynomial $p(x)$ is divisible by $(x - a)^2$, then $p'(x)$ is divisible by $(x - a)$.

13. Let f, g, and h be real-valued functions that are differentiable on an interval I. Prove that the product function $fgh: I \to \mathbb{R}$ is differentiable on I and find $(fgh)'$. ☆

14. Let $f: I \to J$, $g: J \to K$, and $h: K \to \mathbb{R}$, where I, J, and K are intervals. Suppose that f is differentiable at $c \in I$, g is differentiable at $f(c)$, and h is differentiable at $g(f(c))$. Prove that $h \circ (g \circ f)$ is differentiable at c and find the derivative.

15. Suppose that $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are differentiable at $c \in I$ and that $g(c) \neq 0$.

(a) Use Exercise 4(c) and the chain rule [Theorem 1.10] to show that $(1/g)'(c) = -g'(c)/[g(c)]^2$.

(b) Use part (a) and the product rule [Theorem 1.7(c)] to derive the quotient rule [Theorem 1.7(d)].

16. Let I and J be intervals and suppose that the function $f: I \to J$ is twice differentiable on I. That is, the derivative f' exists and is itself differentiable on I. (We denote the derivative of f' by f''.) Suppose also that the function $g: J \to \mathbb{R}$ is twice differentiable on J. Prove that $g \circ f$ is twice differentiable on I and find $(g \circ f)''$.

17. Let $f: I \to \mathbb{R}$, where I is an open interval containing the point c, and let $k \in \mathbb{R}$. Prove the following.

(a) f is differentiable at c with $f'(c) = k$ iff $\lim_{h \to 0} [f(c+h) - f(c)]/h = k$.

*(b) If f is differentiable at c with $f'(c) = k$, then $\lim_{h \to 0} [f(c + h) - f(c - h)]/2h = k$.

(c) If f is differentiable at c with $f'(c) = k$, then $\lim_{n \to \infty} n[f(c + 1/n) - f(c)] = k$.

(d) Find counterexamples to show that the converses of parts (b) and (c) are not true.

$$1-(1+x)^n = n(1+c)^{n-1}(-x) \le -nx,$$

since $0 < 1 + c < 1$ and $n - 1 \ge 0$. It follows that $(1+x)^n \ge 1 + nx$.

2.11 $$f'(x) = \frac{1}{g'(y)} = \frac{1}{ny^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n}x^{1/n-1}$$

2 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the chapter.*

1. Mark each statement True or False. Justify each answer.

(a) A continuous function defined on a bounded interval assumes maximum and minimum values.
(b) If f is continuous on $[a,b]$, then there exists a point $c \in (a,b)$ such that $f'(c) = [f(b)-f(a)]/(b-a)$.
(c) Suppose f is differentiable on (a,b). If $c \in (a,b)$ and $f'(c) = 0$, then $f(c)$ is either the maximum or the minimum of f on (a,b).

2. Mark each statement True or False. Justify each answer.

(a) Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = g'(x)$ for all $x \in (a,b)$, then f and g differ by a constant.
(b) If f is differentiable on (a,b) and $c \in (a,b)$, then f' is continuous at c.
(c) Suppose f is differentiable on an interval I. If f is not injective on I, then there exists a point $c \in I$ such that $f'(c) = 0$.

3. Let $f(x) = x^2 - 4x + 5$ for $x \in [0, 3]$. ☆

(a) Find where f is strictly increasing and where it is strictly decreasing.
(b) Find the maximum and minimum of f on $[0, 3]$.

4. Repeat Exercise 3 for $f(x) = |x^2 - 1|$ on $[0, 2]$.

5. Use the mean value theorem to establish the following inequalities. (You may assume any relevant derivative formulas from calculus.)

(a) $e^x > 1 + x$ for $x > 0$

(b) $\dfrac{x-1}{x} < \ln x < x-1$ for $x > 1$

(c) $7\frac{1}{4} < \sqrt{53} < 7\frac{2}{7}$

(d) $\sqrt{1+x} < 1+\frac{1}{2}x$ for $x > 0$

(e) $\sqrt{1+x} < 5+\dfrac{x-24}{10}$ for $x > 24$

(f) $\sin x \le x$ for $x \ge 0$

(g) $|\cos x - \cos y| \le |x - y|$ for $x, y \in \mathbb{R}$

(h) $x < \tan x$ for $0 < x < \pi/2$

(i) $\arctan x < \dfrac{\pi}{4} + \dfrac{x-1}{2}$ for $x > 1$

(j) $\left|\dfrac{\sin ax - \sin bx}{x}\right| \le |a - b|$ for $x \ne 0$

6. Rolle's theorem requires three conditions be satisfied:
(i) f is continuous on $[a, b]$,
(ii) f is differentiable on (a, b), and
(iii) $f(a) = f(b)$.
Find three functions that satisfy exactly two of these three conditions, but for which the conclusion of Rolle's theorem does not follow. That is, there is no point $c \in (a, b)$ such that $f'(c) = 0$.

7. Let f be continuous on $[a, b]$ and differentiable on (a, b). Prove that for any x and h such that $a \le x < x + h \le b$ there exists an $\alpha \in (0, 1)$ such that $f(x + h) - f(x) = hf'(x + \alpha h)$. ☆

***8.** A function f is said to be **increasing** on an interval I if $x_1 < x_2$ in I implies that $f(x_1) \le f(x_2)$. [For **decreasing**, replace $f(x_1) \le f(x_2)$ by $f(x_1) \ge f(x_2)$.] Suppose that f is differentiable on an interval I. Prove the following:

(a) f is increasing on I iff $f'(x) \ge 0$ for all $x \in I$.
(b) f is decreasing on I iff $f'(x) \le 0$ for all $x \in I$.

9. Show that the converses of parts (a) and (b) of Theorem 2.8 are false by finding counterexamples.

10. Let f be differentiable on $(0, 1)$ and continuous on $[0, 1]$. Suppose that $f(0) = 0$ and that f' is increasing on $(0, 1)$. (See Exercise 8.) Let $g(x) = f(x)/x$ for $x \in (0, 1)$. Prove that g is increasing on $(0, 1)$.

***11.** Let f be differentiable on $[a, b]$. Suppose that $f'(x) \ge 0$ for all $x \in [a, b]$ and that f' is not identically zero on any subinterval of $[a, b]$. Prove that f is strictly increasing on $[a, b]$. ☆

12. Let f be differentiable on $\mathbb{R}$. Suppose that $f(0) = 0$ and that $1 \le f'(x) \le 2$ for all $x \ge 0$. Prove that $x \le f(x) \le 2x$ for all $x \ge 0$.

13. Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 0$, $f(1) = 2$, and $f(2) = 2$. ☆

(a) Show that there exists $c_1 \in (0, 1)$ such that $f'(c_1) = 2$.
(b) Show that there exists $c_2 \in (1, 2)$ such that $f'(c_2) = 0$.
(c) Show that there exists $c_3 \in (0, 2)$ such that $f'(c_3) = \frac{5}{4}$.