Math 4355 – Spring 2010 Name:

TEST #1

No books or notes allowed. Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[3 Pts] (i) State the definition of *orthogonal complement* of an inner product space V.

(ii) Let $V = \mathbb{R}^3$ and consider the subspace of V given by

$$V_0 = \text{span}\,\{(1,0,2), (-1,-1,1)\}.$$

Find the orthogonal complement of V_0 in V.

(2)[3 Pts] Consider the inner product space $V = L^2([0,1])$. Compute the orthogonal projection of the function $f(x) = x^2$, for $x \in [0,1]$, onto the subspace $V_0 = \text{span}\,\{\phi, \psi\}$, where

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(3)[3 Pts] Consider the sequence of functions (f_n) defined by

$$f_n(x) = \begin{cases} nx & 0 \le x < \frac{1}{n} \\ 1 & \frac{1}{n} \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Draw the graph of $f_n(x)$ for two values of n (e.g., $n = 2, 4$). Show that (f_n) converges to the function $f(x) = 1$, $x \in [0,1]$, in the L^2 norm.

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<u>TEST #2</u>

Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[5 Pts] Let $f(x) = \cos^2(x)$.

(a) Sketch a graph of f over the interval $[-\pi, \pi]$.

(b) Expand the function $f(x) = \cos^2(x)$ in a Fourier series valid on the interval $-\pi \le x \le \pi$.

(b) Does the Fourier series of f converge uniformly to f? Justify your answer.

(2)[6 Pts] Consider the function

$$f(x) = \begin{cases} -x^2 & -1 \le x < 0 \\ x^2 & 0 \le x \le 1. \end{cases}$$

(a) Sketch a graph of f over the interval $[-1, 1]$.

(b) Expand the function f in a Fourier series valid on the interval $-1 \le x \le 1$.

[HINT: You can take advantage of the symmetry of f]

(3)[4 Pts] (a) Show that if f is continuous on the interval $0 \le x \le a$, then its even periodic extension is continuous everywhere. Justify your answer.

(b) What about the odd periodic extension? What conditions are necessary to ensure that the odd periodic of f is continuous everywhere? Justify your answer.

[HINT: In both cases, it suffices to check the behavior near $x = 0$]

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TEST #3

Please, write clearly and justify all your steps, to get proper credit for your work. Open book test

(1)[5 Pts] Let

$$f(t) = \begin{cases} t & -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

(a) Compute $\hat{f}$, the Fourier transform of f.
(b) Express the real and imaginary part of $\hat{f}$.
[HINT: You can take advantage of the symmetry of f]

(2)[5 Pts] Let

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

and

$$g(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Compute $h(x) = (\phi * g)(x)$.
(b) Sketch the graphs of ϕ, f and h over the interval $[-1, 3]$.

(3)[3 Pts] Consider the filter

$$f \to f * h_d,$$

where

$$h_d(t) = \begin{cases} 1/d & 0 \le t < d \\ 0 & \text{otherwise} \end{cases}.$$

Let

$$f(t) = e^{-t}\,(\cos 2t + \sin 3t + \cos 18t + \sin 90t)\,, \quad t \in [0, 2\pi].$$

Which value(s) of the parameter d for the filter h_d will ensure that the components of the signal f with frequencies above 50 are removed and the frequencies in the range 0 to 18 are retained? Justify your answer.

TEST # 1

SOLUTION

① (i) The ORTHOGONAL COMPLEMENT of V_0 in V is the set of all vectors in V which are orthogonal to V_0:

$$V_0^{\perp} = \{ v \in V : \langle v, w \rangle = 0 \quad \forall w \in V_0 \}$$ 1 Pt

(ii) Since $\mathbb{R}^3$ is 3-dimensional and V_0 is 2-dimensional, $V_0^{\perp}$ is the span of the vectors which are orthogonal to $(1,0,2)$ and $(-1,-1,1)$.

$$(1,0,2)\times(-1,-1,1) = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ -1 & -1 & 1 \end{vmatrix} = (2,-3,-1)$$ 1 Pt

Hence

$$V_0^{\perp} = \{ v \in V : \quad v = \alpha (2,-3,-1), \quad \alpha \in \mathbb{R} \}$$ 1 Pt

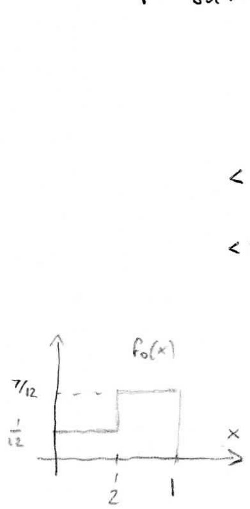
② φ and ψ are an ONB of V_0. Hence, f_0, the orthogonal projection of f onto V_0 is given by

$$f_0(x) = \langle f, \varphi \rangle \varphi(x) + \langle f, \psi \rangle \psi(x)$$ 1 Pt

$$\langle f, \varphi \rangle = \int_0^1 x^2 \varphi(x)\,dx = 1/3$$ 1 Pt

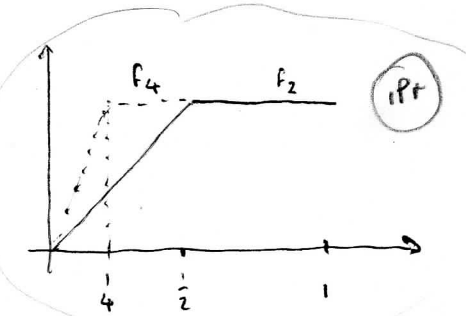
$$\langle f, \psi \rangle = \int_0^1 x^2 \psi(x)\,dx = \int_0^{1/2} x^2\,dx - \int_{\frac{1}{2}}^1 x^2\,dx = \frac{x^3}{3}\Big|_0^{1/2} - \frac{x^3}{3}\Big|_{\frac{1}{2}}^1 = \frac{(1/2)^3}{3} - \frac{1}{3} + \frac{(1/2)^3}{3}$$ 1 Pt

$$= -\frac{1}{4}$$



$$f_0(x) = \frac{1}{3}\varphi(x) - \frac{1}{4}\psi(x)$$

③



Need to show that $f_n(x) \to f(x)$ in L^2 norm

That is $\| f_n - f \|^2 \to 0$

That is

$$\int_0^1 (f_n(x) - 1)^2\,dx = \int_0^{\frac{1}{n}} (nx-1)^2\,dx =$$

$$= \int_0^{\frac{1}{n}} (1 - 2nx + n^2x^2)\,dx = x - nx^2 + n^2\frac{x^3}{3}\Big|_0^{\frac{1}{n}} = \frac{1}{3n} \to 0 \text{ as } n$$

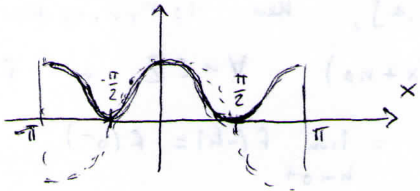
2 Pt

This shows that $\lim_{n\to\infty} \| f_n - f \| = \lim_{n\to\infty} \frac{1}{3n} = 0$

TEST #2 SOLUTION

$f(x) = \cos^2 x$

(a)



[1 Pt] for graph

Notice that $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ ($\cos^2 x$ is π-periodic)

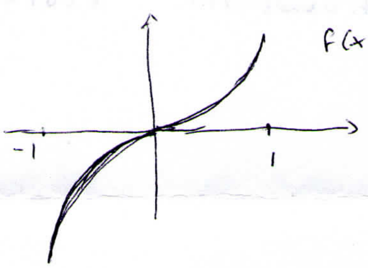
(b) By the uniqueness of the Fourier series, it follows that the Fourier series of f, over the interval $[-\pi, \pi]$ is

$$F(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$$

[3 Pts]

(c) Since f is continuous and piecewise smooth, then $F(x)$ converges uniformly to f on $[-\pi, \pi]$ 1 Pt

) (a)



Notice that f is an odd function

[1 Pt] for gr

(b) Since f is an odd function, its Fourier series only contain the sin-terms. Hence it can be computed as

$$F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \qquad \text{where} \qquad b_k = 2\int_0^1 f(x)\sin(k\pi x)\, dx$$

$$b_k = 2\int_0^1 x^2 \sin k\pi x \, dx = 2\left[-\frac{1}{k\pi}x^2\cos k\pi x\Big|_0^1 + \frac{2}{k\pi}\int_0^1 x\cos k\pi x\, dx\right]$$

$$= 2\left[-\frac{x^2}{k\pi}\cos k\pi x\Big|_0^1 + \cancel{\frac{2}{k^2\pi^2}x\sin k\pi x}\Big|_0^1 - \frac{2}{k^2\pi^2}\int_0^1 \sin k\pi x\, dx\right]$$

$$= 2\left[-\frac{x^2}{k\pi}\cos k\pi x\Big|_0^1 + \cancel{\frac{2}{k^2\pi^2}x\sin k\pi x}\Big|_0^1 + \frac{2}{k^3\pi^3}\cos k\pi x\Big|_0^1\right]$$

$$= 2\left[-\frac{1}{k\pi}\cos k\pi + \cancel{\frac{2}{k^2\pi^2}\sin k\pi} + \frac{2}{k^3\pi^3}\cos k\pi - \frac{2}{k^3\pi^3}\right]$$

← THI is AL to get FULL

$$= 2\left(-\frac{(-1)^k}{k\pi} + \frac{2}{(k\pi)^3}\left((-1)^k - 1\right)\right)$$

Pts]

3 (a) If f is continuous on $[0,a]$, then its even periodic extension satisfies $f(x) = f(x+na)$ $\forall n \in \mathbb{Z}$ and $f(x) = f(-x)$

[.pts]

In particular, $f(0^+) = \lim_{h\to 0^+} f(h) = \lim_{h\to 0^+} f(-h) = f(0^-)$

That is the periodic extension is continuous at $x=0$

(b) The odd extension satisfies $f(x) = -f(-x)$

Hence, in general $\lim_{h\to 0^+} f(h) \neq \lim_{h\to 0^+} f(-h) = \lim_{h\to 0^+} -f(h)$

[2Pts]

The two sides are the same iff ~~f(h) = f(h)~~,

$\lim_{h\to 0^+} f(h) = -\lim_{h\to 0} f(h) = 0$

That is, the odd extension of f is continuous iff $f(0) = 0$

TEST #3 SOLUTIONS

① (a) $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} t\, e^{-i\omega t}\, dt$$

NOTICE :

$t e^{-i\omega t} = t\cos\omega t - i t \sin\omega t$

Since $t\cos\omega t$ is ODD, it gives no contribution to $\hat{f}(\omega)$

[4Pts]

$$= \frac{2i}{\sqrt{2\pi}} \int_{0}^{\frac{1}{2}} t\,(-\sin\omega t)\, dt$$

$$= \frac{2i}{\sqrt{2\pi}} \left[\frac{t}{\omega}\cos\omega t \Big|_0^{\frac{1}{2}} - \frac{1}{\omega}\int_0^{\frac{1}{2}} \cos\omega t \right] =$$

$$= \frac{2i}{\sqrt{2\pi}} \left[\frac{1}{2\omega}\cos\left(\omega\frac{1}{2}\right) - \frac{1}{\omega^2}\sin\frac{\omega}{2} \right]$$

(b) $\hat{f}(\omega)$ is PURELY IMAGINARY $\quad \hat{f}(\omega) = i\, \mathrm{Im}\left[\hat{f}(\omega)\right]$

[1Pt] $\mathrm{Re}\, \hat{f}(\omega) = 0$

③ The convolution acts as local averaging operator on windows of size d

To remove frequencies above 50, need

$$d \geq \frac{2\pi}{50} \simeq 0.12$$

[3Pts] To preserve frequencies below 18, need

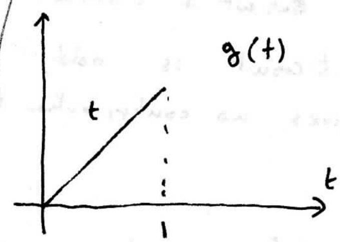
$$d \ll \frac{2\pi}{18} \simeq 0.33$$

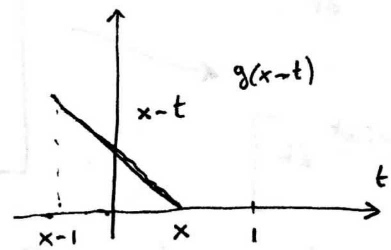
For example, we can choose $d \simeq 0.15$

(2) $$h(x) = (\varphi * g)(x) = \int_{\mathbb{R}} \varphi(t)\, g(x-t)\, dt$$

(a)

$$= \int_0^1 g(x-t)\, dt$$





[2Pts]

It is clear from the graph of $g(x-t)$ that $h(x) = 0$ if $x < 0$

and $h(x) = 0$ if $x - 1 > 1 \iff x > 2$

Hence, we only need to examine $0 \le x \le 2$

If $0 \le x \le 1$, then $h(x) = \int_0^x (x-t)\,dt = \left(xt - \frac{t^2}{2}\right)\Big|_0^x = x^2 - \frac{x^2}{2} = \frac{x^2}{2}$

[2Pts]

If $1 < x < 2$, then $h(x) = \int_{x-1}^1 (x-t)\,dt = \left(xt - \frac{t^2}{2}\right)\Big|_{x-1}^1 = x - \frac{1}{2} - (x-1)x + \frac{(x-1)^2}{2}$

$$= x\left(1 - \frac{x}{2}\right)$$

Conclusion

$$h(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0 & \text{if } x > 2 \\ \frac{x^2}{2} & \text{if } 0 \le x \le 1 \\ x\left(1 - \frac{x}{2}\right) & \text{if } 1 < x \le 2 \end{cases}$$

(b)

[1Pt]