Metamorphosis of Images in Reproducing Kernel Hilbert Spaces

by Heng Zhao

May 6, 2020

Content

- What is diffeomorphic matching of shapes?
- How to build diffeomorphisms
- Large Deformation Diffeomorphic Metric Mapping.
- Shooting Method.
- Numerical Experiments.
- Results on MNIST
- Conclusion

What is diffeomorphic matching of shapes?

- Metamorphosis is a method for diffeomorphic matching of shapes, with many potential applications for anatomical shape comparison in medical image analysis; it is a central problem in the field of computational anatomy.
- In applications, we always need to compare the similarity of two different shapes; diffeomorphic matching will provide a way to compare shapes. Two shapes are similar if one can be obtained from the other via a small deformation.

How to build diffeomorphisms

- For $t \in [0,1]$, velocity field $v(t) : \mathbb{R}^d \to \mathbb{R}^d$. The position $x(t) \in \mathbb{R}$ at time t of a particle that moves along this velocity field is described by

$$\frac{\mathrm{d}x}{\mathrm{d}t}(x) = v\left(t, x\left(t\right)\right). \tag{1}$$

- This is a deformation of the space at time t, denoted $\varphi\left(t\right)$, so $\varphi\left(t, x\right)$ is the position of particle at time t started its motion at x at time 0. In particular, $\varphi\left(0, x\right) = x$.
- As long as we take v(t) "very regular" with respect to the space variables, the transformation will be a diffeomorphism: it will map smooth curves onto smooth curves, corners onto corners, and preserve presence or lack of self-intersection points.

How to build diffeomorphisms

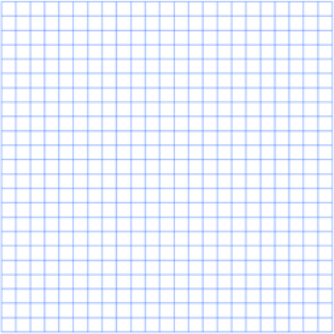

Figure: Initial Grid.

How to build diffeomorphisms

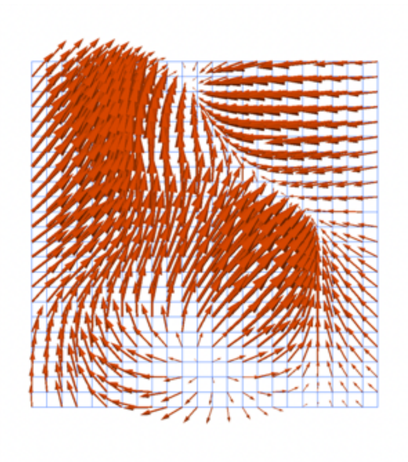

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

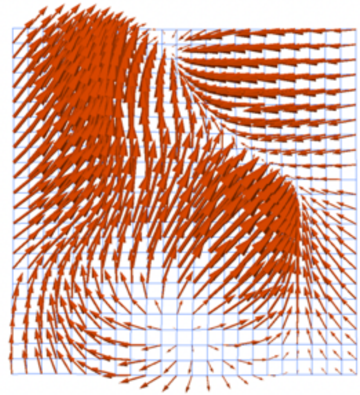

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

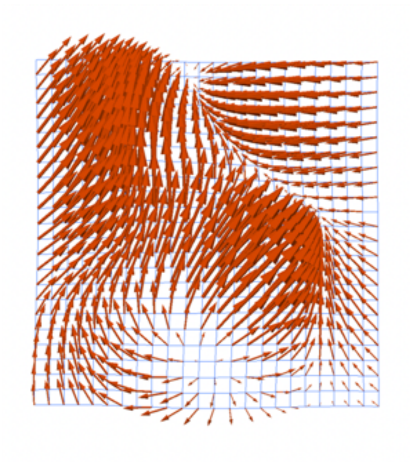

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

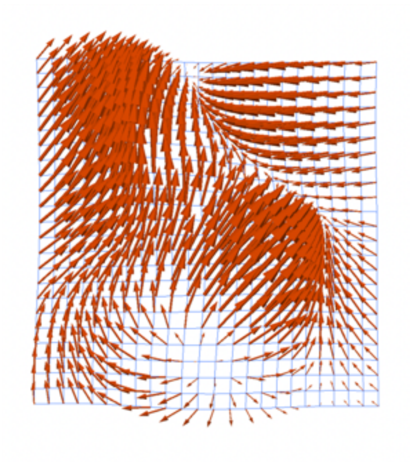

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

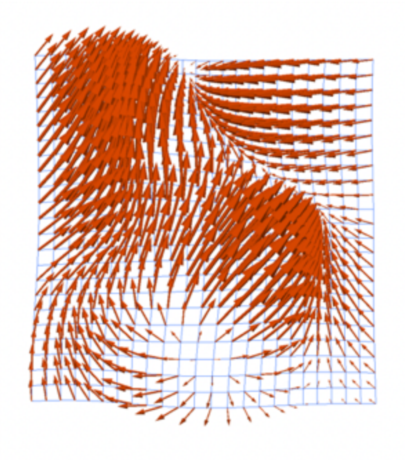

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

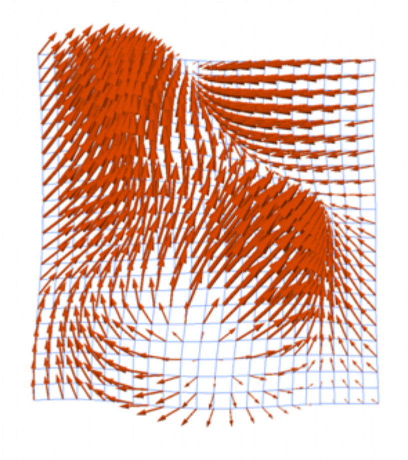

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

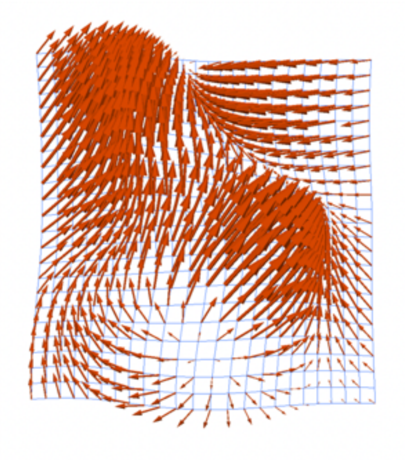

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

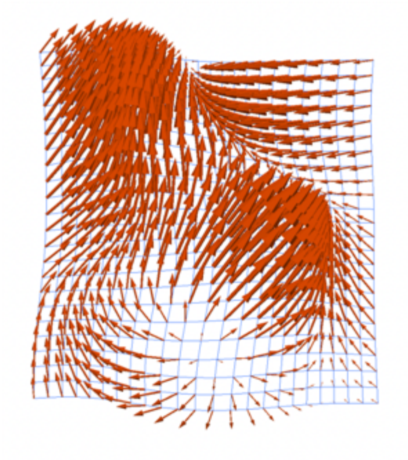

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

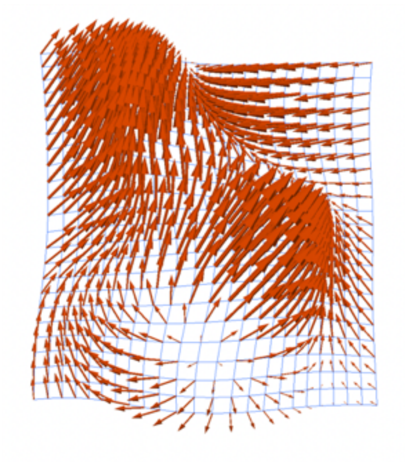

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

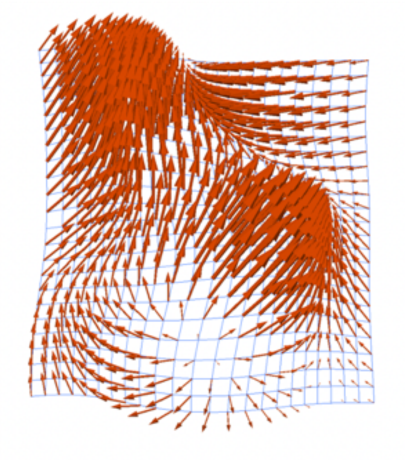

Figure: A controller specifies a direction at every point, at every time.

How to build diffeomorphisms

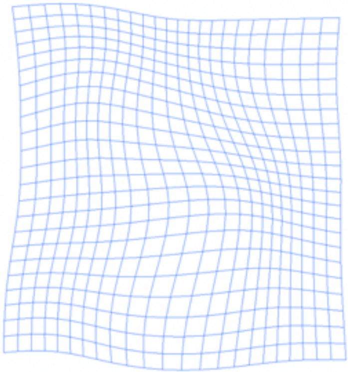

Figure: Final Grid.

Large Deformation Diffeomorphic Metric Mapping

- Fix a shape q_0, the template, from which we want to register another shape q_1 (the target).
- A time-dependent velocity field $(t, x) \to v(t, x)$ yields a deformation $(t, x) \to \varphi(t, x)$, which acts onto q_0 as denoted by $q(t) := \varphi(t) \cdot q_0$. The goal is now to find v^* which minimizes a functional

$$J(v) = \frac{1}{2} \int_0^1 \|v(t)\|_V^2 \, dt + U(q(1)) \to min \tag{2}$$

subject to $\partial_t q(t) = v(t) \cdot q(t)$, $q(0) = q$

- where $\|\cdot\|_V$ is an appropriate Hilbert norm. The data attachment U(q(1)) is a crude measure of the difference between the deformed shape q(1) and the target q_1.

Large Deformation Diffeomorphic Metric Mapping

- In order to connect two images $q(0)$ and $q(1)$ in H with a continuous path $q(t)$, image metamorphosis solves the optimal control problem(3):

$$\frac{1}{2}\int_0^1 \|v(t)\|_V^2 dt + \frac{1}{2\sigma^2}\int_0^1 \|\zeta(t)\|_H^2 dt \longrightarrow \min$$

$$\text{subject to } \dot{q}(t) = \nabla q(t) \cdot v(t) + \zeta(t),\ q(0) = q^{(0)} \text{ and } q(1) = q^{(1)}.$$

Large Deformation Diffeomorphic Metric Mapping

- Well known results on RKHS imply that these optimal solutions must be of the form

$$v(t,\cdot) = \sum_{\ell=1}^{N} K_V(\cdot, x_\ell(t)) z_\ell(t).$$

$$\zeta(t,\cdot) = \sigma^2 \sum_{\ell=1}^{N} K_H(\cdot, x_\ell(t)) \alpha_\ell.$$

Large Deformation Diffeomorphic Metric Mapping

- for some coefficients z and α, and that their norms are given by

$$\|v\|_V^2 = \sum_{k,\ell=1}^{N} z_k(t) \cdot K_V(x_k(t), x_\ell(t)) z_\ell(t)$$

$$\|\zeta(t)\|_H^2 = \sum_{k,\ell=1}^{N} K_H(x_k(t), x_\ell(t)) \alpha_k(t) \alpha_\ell(t).$$

Large Deformation Diffeomorphic Metric Mapping

- Solutions of (3) are therefore solutions of the reduced problem (4):

$$\frac{1}{2}\sum_{k,\ell=1}^{N}\int_0^1 z_k(t)\cdot K_V(x_k(t),x_\ell(t))z_\ell(t)dt + \frac{1}{2\sigma^2}\sum_{k,\ell=1}^{N}\int_0^1 K_H(x_k(t),x_\ell(t))\alpha_k(t)\alpha_\ell(t)dt \longrightarrow \min \tag{11}$$

subject to

$$\dot{x}_k(t) = \sum_{\ell=1}^{N} K_V(x_k(t),x_\ell(t))z_\ell(t),$$

$$\dot{m}_k(t) = \sum_{\ell=1}^{N} K_H(x_k(t),x_\ell(t))\alpha_\ell(t),$$

$$m_k(0) = q^{(0)}(x_k^{(0)}) \text{ and } m_k(1) = q^{(1)}(x_k(1)).$$

Shooting Method

- Then this paper derives the shooting method for (4)

Algorithm 1 Shooting Algorithm

Require: template $q^{(0)}$, target $q^{(1)}$; specify kernels K_V, K_H; matching parameter σ

$\alpha \leftarrow 0, z^{(0)} \leftarrow 0$

while (not stop CG) **do**

1. Compute $\partial_{z^{(0)}} E, \partial_\alpha E$:

   1.1 Compute $dE = \partial_{x_k} E \; dx_k + \partial_{m_k} E \; dm_k$ given by (14)

   1.2 Compute $\xi_z(0), \eta(0)$: solve the adjoint system backwards in time starting from dE at $t = 1$.

2. Update conjugate direction and perform line search

3. Update $z^{(0)}, \alpha$

end while

Numerical Experiments

- For numerical experiment, they use

$$K_V(x,y) = (1 + u + 3u^2/7 + 2u^3/21 + u^4/105)\, e^{-u}.\mathrm{Id}_{\mathbb{R}^d}$$

$$K_H(x,y) = (1 + \tilde{u} + \tilde{u}^2/3)\, e^{-\tilde{u}}$$

- with $u = |x - y| / \tau_V$ and $\tilde{u} = |x - y| / \tau_H$, where τ_V and τ_H are width parameters associated to the reproducing kernels1.
- In this numerical experiment, they use $\tau_V = 1.5$ and $\tau_H = 0.5$

Results

Figure: Morphing of letter D from MNIST training set: top row shows evolution of the template; bottom row shows evolution of deformed template.

Conclusion

- In this paper, they proposed a particle based optimization method for their estimation and a shooting method based on a specific reproducing kernels.
  This algorithm allows for numerically stable sparse representation of the target image in a template-centered coordinate system, which is hard to obtain using former methods. The introduction of the Sobolev Space norm for the images plays a critical role as it allows for particle solutions that would not be possible using an L^2 norm.