HW #1

(1) Let $K \subseteq \mathbb{R}$ be compact and $U \supseteq K$ be open. Let

$$d = \text{dist}(K, \mathbb{R} \backslash U) = \inf\{|x - y| : x \in K, y \notin U\}$$

and set

$$V = \{y \in \mathbb{R} : \text{dist}(y, K) < d/3\}.$$

Let k be any function in $C_c^\infty(\mathbb{R})$ such that $\int k = 1$, $k(x) = 0$ for $|x| > d/3$. Prove that the convolution $g = \chi_V * k$ has the following properties

(a) $g \in C_c^\infty(\mathbb{R})$
(b) $0 \le g \le 1$.
(c) $g(x) = 1$ for $x \in K$; $g(x) = 0$ if $x \notin U$.

(2) Consider the following problems about convolution.

(a) Show that if $f \in L^1(\mathbb{R})$ and $f = f * f$, then it must be $f = 0$ a.e.

(b) Show that there exist nontrivial functions $f \in L^2(\mathbb{R})$ such that $f = f * f$.

(3) Show that if if $f, \hat{f} \in L^1(\mathbb{R})$ then $f \in L^p(\mathbb{R})$, any $1 \le p \le \infty$.

(4) Show that, if $f \in L^2(\mathbb{R})$, then $\{T_k f : k \in \mathbb{Z}\}$ is an orthonormal set if and only if $\sum_{k \in \mathbb{Z}} |\hat{f}(\xi - k)|^2 = 1$ a.e.