HW #1 SOLUTION

(1) [assume $K \geq 0$]

Since $\overline{K}$ is compact and $\mathbb{R} \setminus U$ is closed, then $d = \text{dist}(\overline{K}, \mathbb{R}\setminus U) > 0$

Set $\vartheta = \kappa * \chi_V$

Since κ, χ_V have compact support, so does ϑ (by properties of convolution)

Since $\kappa \in C^\infty$, $\chi_V \in L^1$, so is $\vartheta \in C^\infty$ (by properties of convolution)

$$\vartheta(x) = \int_V \kappa(x-y)\,dy \leq \int_{\mathbb{R}} \kappa(y)\,dy = 1 \qquad \text{(here we use } \kappa \geq 0)$$

This shows that $0 \leq \vartheta(x) \leq 1$

If $x \in \overline{K}$, $y \notin V$, then $|x-y| \geq d/3$

Here, for $x \in \overline{K}$,

$$\vartheta(x) = \int_V \kappa(x-y)\,dy = \int_{\mathbb{R}} \kappa(x-y)\,dy = 1$$

If $x \in \overline{U}$, $y \in V$, let x_0 be any point in $\overline{K}$

$$|x - x_0| \leq |x-y| + |y-x_0|$$

Then $|x-y| \geq |x-x_0| - |y-x_0|$

$$\geq d - \frac{d}{3} = \frac{2d}{3} > \frac{d}{3}$$

This implies that

$$\vartheta(x) = \int_V \kappa(x-y)\,dy = 0$$

(2) - If $f \in L^1$, then $\hat{f}$ is continuous.

$$f = f * f \Rightarrow \hat{f}(\xi) = \hat{f}(\xi)^2 \quad \forall \xi \in \mathbb{R}$$

This implies that $\hat{f}(\xi) \in \{0, 1\}$

Since $\hat{f}$ is continuous, either $\hat{f}(\xi) \equiv 0$ or $\hat{f}(\xi) \equiv 1$.

The last situation is impossible by Riemann-Lebesgue Theorem.

- Let $f \in L^2$, $f = f * f \Rightarrow \hat{f} = \hat{f}^2$

Set $\hat{f}(\xi) = \chi_{(-\frac{1}{2}, \frac{1}{2})}(\xi)$ $\qquad f(x) = \text{sinc}(\pi x)$

In this case $\hat{f} = \hat{f}^2$ and $f = f * f$

(3) Since $f, \hat{f} \in L^1$, then $f, \hat{f} \in C_0 \subset L^\infty$

Hence, for any p s.t. $1 \le p \le \infty$,

$$|f(x)|^p = |f(x)|^{p-1}\,|f(x)| \le \|f\|_\infty^{p-1}\,|f(x)|$$

Hence

$$\left(\int |f(x)|^p\,dx\right)^{1/p} \le \|f\|_\infty^{\frac{p-1}{p}} \left(\int |f(x)|\,dx\right)^{1/p} < \infty$$

(4) $\{T_k f : k \in \mathbb{Z}\}$ is ON set

$$\Leftrightarrow \quad \langle T_k f, T_\ell f\rangle = \delta_{k,\ell} = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k \ne \ell \end{cases}$$

$$\begin{aligned}
\langle T_k f, T_\ell f\rangle &= \langle T_{k-\ell} f, f\rangle = \\
&= \int_{\mathbb{R}} (T_{k-\ell} f)^\wedge\, \overline{\hat{f}} \qquad \text{by Plancherel} \\
&= \int_{\mathbb{R}} e^{-2\pi i \xi (k-\ell)} |\hat{f}(\xi)|^2\, d\xi \qquad \text{by periodicity} \\
&= \int_0^1 e^{-2\pi i \xi (k-\ell)} \sum_{n \in \mathbb{Z}} |\hat{f}(\xi + n)|^2\, d\xi
\end{aligned}$$

Set $G(\xi) = \sum_{n \in \mathbb{Z}} |\hat{f}(\xi + n)|^2$

Since $\int_0^1 G(\xi)\,d\xi = \int |\hat{f}(\xi)|^2\,d\xi$ then $G \in L^1([0,1])$

The computation shows that $\langle T_k f, T_\ell f\rangle$ is the $(k-\ell)$-Fourier coefficient of the function ~~G~~ $G \in L^1([0,1])$.

In other words

$$\left.\begin{aligned} \langle G, e_m\rangle &= 0 \quad \forall m \ne 0 \\ \langle G, e_0\rangle &= 1 \end{aligned}\right\} \Leftrightarrow \{T_k f\} \text{ ON set}$$

By the uniqueness of Fourier series this implies that

$$G(\xi) = \sum_{n \in \mathbb{Z}} |\hat{f}(\xi + n)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R} \Leftrightarrow \{T_k f\} \text{ ON set}$$

Hence $\{T_k f\}$ ON set $\Leftrightarrow$ $G(\xi) = 1$ a.e. $\xi \in \mathbb{R}$