Math 6397 – Fall 2017 Name:

HW #2

(1) (a) Given $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in C_c^\infty(\mathbb{R})$, prove that convolution commutes with differentiation, that is

$$\phi * f' = \phi' * f = (\phi * f)'$$

(b) Given $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in C^\infty(\mathbb{R})$, prove the product rule

$$(\phi f)' = \phi' f + \phi f'$$

.

(c) Formulate the analogous result for $\mathcal{S}'$.

(d) Verify that $\phi' * h = \phi * h' = \phi * \delta = \phi$, where h is the Heaviside function.

(2) Show that $x\delta'(x) = -\delta(x)$

(3) Suppose f is a piecewise C^1 function on $\mathbb{R}$ that is differentiable everywhere except at t_0 and has a jump discontinuity at t_0. Show that $f' = f^{(1)} + (f(t_0^+) - f(t_0^-))\,\delta$, where f' denotes the distributional derivative and $f^{(1)}$ denotes the pointwise derivatve (valid for all $t \neq t_0$).

(4) Let $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$. Show that , for some $N \geq 0$, $|\partial^\alpha (f * \phi)(x)| \leq C_\alpha\,(1 + |x|)^N$ for all α. To prove this statement, proceed as follows:

(a) For any $x, y \in \mathbb{R}^n$ and any $k \geq 0$, show that $|y|^k \leq (1 + |x - y|)^k (1 + |x|)^k$.

(b) $\sup_y |y^\beta \partial^\alpha \phi(x - y)| \leq C_{\alpha\beta}(1 + |x|)^\beta$.

(c) $|(f * \phi)(x)| \leq C\,(1 + |x|)^N$ for some $N \geq 0$.

(d) Repeat the argument with ϕ replaced by $\partial^\alpha \phi$ to obtain the desired estimate.

(5) Given $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ is

$$H^s(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) : \, (1 + |\xi|)^s\, \hat{f}(\xi) \in L^2(\mathbb{R})\}.$$

(a) Show that $H^s(\mathbb{R}$ is a Hilbert space with inner product

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, (1 + |\xi|)^{2s}\, d\xi$$

(b) The Fourier transform is a unitary map of $H^s(\mathbb{R})$ onto $L_s^2(\mathbb{R}) = \{f \in \mathcal{S}' : \, \|f\|_{2,s} = \left(\int |f(x)|^2 (1 + |x|)^{2s}\, dx\right)^{1/2} < \infty\}$.

(c) $\mathcal{S}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$.