HW #2

① (a) Let $f \in \mathcal{D}'$, $\varphi \in C_c^\infty$. For any $\psi \in C_c^\infty$,

$$(f*\varphi)'[\psi] = -(f*\varphi)[\psi'] = -f[\psi' * \tilde{\varphi}]$$
$$(f'*\varphi)[\psi] = f'[\psi * \tilde{\varphi}] = -f[(\psi*\tilde{\varphi})'] = -f[\psi' * \tilde{\varphi}]$$
$$(f*\varphi')[\psi] = f[\psi * \tilde{\varphi'}] = -f[\psi * (\tilde{\varphi})'] = -f[\psi' * \tilde{\varphi}]$$

(b) Let $f \in \mathcal{D}'$, $\varphi \in C^\infty$. For any $\psi \in C_c^\infty$,

$$(\varphi f)'[\psi] = -(\varphi f)[\psi'] = -f[\psi'\varphi] = -f[(\psi\varphi)' - \varphi'\psi] =$$
$$= -f[(\psi\varphi)'] + f[\varphi'\psi] = f'[\psi\varphi] + \varphi' f[\psi]$$
$$= \varphi f'[\psi] + \varphi' f[\psi]$$

(c) Property in (b) holds also if $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$. However, we need to assume that φ grows at most polynomially, to be able to carry out the proof above.

(d) Recall that $h'[\psi] = \delta[\psi]$ for any $\psi \in C_c^\infty$.

Hence, by (a) $\quad \varphi' * h = \varphi * h' = \varphi * \delta = \varphi$

② For any $\varphi \in \mathcal{D}$,

$$x\delta'[\varphi] = \delta'[x\varphi] = -\delta[(x\varphi)'] = -\delta[\varphi + x\varphi'] =$$
$$= -\delta[\varphi] - \cancel{\delta[x\varphi']}$$

In fact $\quad \delta[x\varphi'] = x\varphi'(x)\big|_{x=0} = 0$

③ Since $f \in C^1$ has a jump discontinuity at t_0, $\lim_{y \to t_0^-} f(y) = f(t_0^-)$ and $\lim_{y\to t_0^+} f(y) = f(t_0^+)$, where $f(t_0^-)$, $f(t_0^+)$ exist.

For $\varphi \in C_c^\infty$,

$$f'[\varphi] = -f[\varphi'] = -\int f(x)\varphi'(x)\,dx =$$
$$= -\int_{-\infty}^{t_0} f(x)\varphi'(x)\,dx - \int_{t_0}^{\infty} f(x)\varphi'(x)\,dx$$

int. by parts

$$= -f(x)\varphi(x)\Big|_{-\infty}^{t_0^-} + \int_{-\infty}^{t_0} \varphi(x) f^{(1)}(x)\,dx +$$
$$- f(x)\varphi(x)\Big|_{t_0^+}^{\infty} + \int_{t_0}^{\infty} \varphi(x) f^{(1)}(x)\,dx =$$
$$= \int_{-\infty}^{\infty} f^{(1)}(x)\varphi(x)\,dx + (f(t_0^+) - f(t_0^-))\,\underbrace{\varphi(t_0)}_{T_{t_0}\delta[\varphi]}$$
$$= f^{(1)}[\varphi] + (f(t_0^+) - f(t_0^-))\,T_{t_0}\delta[\varphi]$$

④ Let $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$

(a) For any $x, y \in \mathbb{R}^d$, $k \geq 0$,

$|y| = |y + x - x| \leq |y - x| + |x| \leq (1+|x|) + |x-y| + |x-y||x| = (1+|x|) + (1+|x|)|x-y|$

Hence $|y| \leq (1+|x|)(\cancel{1+|x-y|})$, $|y|^k \leq (1+|x|)^k (1+|x-y|)^k$

(b) By part (a), $\cancel{\sup (1+|y|)^k}$ $|y|^k |\partial^\alpha \varphi(x-y)| \leq (1+|x-y|)^k (1+|x|)^k |\partial^\alpha \varphi(x-y)|$

Hence $\sup_y |y|^k |\partial^\alpha \varphi(x-y)| \leq \left(\sup_y (1+|x-y|)^k |\partial^\alpha \varphi(x-y)|\right)(1+|x|)^k$

Since $\varphi \in \mathcal{S}$, $\sup_y (1+|x-y|)^k |\partial^\alpha \varphi(x-y)| = C_{\alpha k} < \infty$

(c) $|f * \varphi(x)| = \cancel{\left|\int f(y)\varphi(x-y)\,dy\right| \leq \int |f(y)||\varphi(x-y)|\,dy}$. Since $f \in \mathcal{S}'$, by def. of continuity in $\mathcal{S}$,

$= |f[T_x \tilde{\varphi}]|$ ~~we have that~~:

$\leq C \sum_{|\alpha| \leq k} \sup_y (|y|)^N |\partial^\alpha \varphi(x-y)|$

$\leq C \sum_{|\alpha| \leq k} C_{N,\alpha} (1+|x|)^N \leq c(1+|x|)^N$

(d) $|\partial^\alpha (f*\varphi)(x)| = |f * \partial \varphi(x)| = |f[T_x \partial^\alpha \varphi]|$

~~As~~: $\leq C \sum_{|\alpha'| \leq k} \sup_y |y|^N |\partial^{\alpha+\alpha'} \varphi(x-y)|$

$\leq C \sum_{|\alpha'| \leq k} C_{N,\alpha+\alpha'} (1+|x|)^N \leq C_\alpha (1+|x|)^N$

⑤ (a) For $s \geq 0$, let $L^2_s = \{ f \in \mathcal{S}' : \|f\|_{L^2_s} = \int |f(x)|^2 (1+|x|)^{2s}\,dx < \infty \}$

This is a Banach space.

It is easy to verify that $\langle f, h\rangle_{HS} = \int \hat{f}(\xi)\hat{g}(\xi)(1+|\xi|)^{2s}\,d\xi$ is an INNER PRODUCT

We will show below that H^s is isometrically isomorphic to L^2_s. Hence H^s is a complete INNER PRODUCT SPACE, that is, a Hilbert space.

(b) $\mathcal{F}$ is linear. To show that $\mathcal{F}: H^s \to L^2_s$ is an isometry, observe that, for any $f, h \in H_s$

$\langle f, h\rangle_{H_s} = \langle \hat{f}, \hat{h}\rangle_{L^2_s} = \int \hat{f}(\xi)\hat{h}(\xi)(1+|\xi|)^{2s}\,d\xi$

To show that the map is also surjective, note that, for any $g \in L^2_s$ we have that $f = \mathcal{F}^{-1} g \in H^s$. Now $\mathcal{F} f = g$.

This shows that $\mathcal{F}: H^s \to L^2_s$ is a UNITARY MAP.

(c) $\mathcal{S}$ is dense in the weighted L^2 space L^2_s. Since L^2_s and H_s are isometrically isomorphic, then $\mathcal{S}$ is also dense in H^s. Specifically, given any $f \in H_s$, we can find a sequence $(\varphi_n) \subset \mathcal{S}$ s.t. $\lim_n \|\varphi_n - \hat{f}\|_{L^2_s} = 0$

Hence there is a sequence $(\check{\varphi}_n) \subset \mathcal{S}$ s.t. $\lim_n \|\check{\varphi}_n - f\|_{H_s} = \lim_n \|\varphi_n - \hat{f}\|_{L^2_s} = 0$