Math 6397 – Fall 2017 Name:

HW #3

(1) Let $\mathbb{H}$ be Hilbert space and $\{e_j : j = 1, 2, \dots\} \subset \mathbb{H}$.

(a) Prove that

$$\|f\|^2 = \sum_j |\langle f, e_j\rangle|^2, \quad \text{for all } f \in \mathbb{H}$$

if and only if

$$f = \sum_j \langle f, e_j\rangle\, e_j \quad \text{for all } f \in \mathbb{H}$$

where the last sum converges in $\mathbb{H}$.

(b) Suppose that $\|f\|^2 = \sum_j |\langle f, e_j\rangle|^2$ holds for all f belonging to a dense subspace of $\mathbb{H}$. Prove that the same equality holds for all $f \in \mathbb{H}$.

(2) Let $\mathbb{H}$ be Hilbert space and $\{e_j : j = 1, 2, \dots\} \subset \mathbb{H}$ be a Parseval frame, that is

$$\|f\|^2 = \sum_j |\langle f, e_j\rangle|^2, \quad \text{for all } f \in \mathbb{H}.$$

Show that if $\|e_j\| = 1$ for all j, then $\{e_j : j = 1, 2, \dots\}$ is an orthonormal basis of $\mathbb{H}$.

(3) Suppose that $\{\psi_{j,k} = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is an ON wavelet system where $\psi \in L^2(\mathbb{R})$ and $\hat{\psi}$ is continuous at zero. Prove that it must be $\hat{\psi}(0) = 0$. (Hint: use the fact that $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$).

(4) Suppose $\psi \in L^2(\mathbb{R})$ is such that $\hat{\psi}(\xi) = \chi_{[-1,1]\setminus[-1/2,1/2]}(\xi)$.

(a) Prove that $\{\psi_{j,k} = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is an ON wavelet system.

(b) Extend the construction to dimension $d = 2$.