HW 3 SOLUTION

1(a) $$\|f\|^2 = \langle f,f\rangle = \lim_{N\to\infty}\sum_{j=1}^{N} \langle f,e_j\rangle\langle e_j,f\rangle = \sum_{j=1}^{\infty} |\langle f,e_j\rangle|^2 \qquad \text{(i)}$$

(⇒) Suppose the last equality holds.

Let $S_N = \sum_{j=1}^{N} \langle f,e_j\rangle e_j$. We will show it is a Cauchy sq. Let $1 \le n \le N$.

$$\|S_M - S_N\| = \Big\|\sum_{j=M}^{N} \langle f,e_j\rangle e_j\Big\| = \sup_{\|g\|\le 1} \Big|\Big\langle \sum_{j=n}^{N} \langle f,e_j\rangle e_j, g\Big\rangle\Big|$$

$$\le \sup_{\|g\|\le 1} \sum_{j=n}^{N} |\langle f,e_j\rangle||\langle g,e_j\rangle| \le \Big(\sum_{j=n}^{N} |\langle f,e_j\rangle|^2\Big)^{1/2}$$

Since, $\sum_{j=n}^{N} |\langle g,e_j\rangle|^2 \le \|g\|^2 = 1$

Since $\sum_{j=1}^{\infty} |\langle f,e_j\rangle|^2 = \|f\|^2 < \infty$, it follows that $\lim_{\substack{n\to\infty \\ n\le N}} \sum_{j=n}^{N} |\langle f,e_j\rangle|^2 = 0$

Thus (S_N) is Cauchy and $\lim S_N = h = \sum_{j=1}^{\infty} \langle f,e_j\rangle e_j \in \mathcal{H}$

By polarization identity applied to (i), we have that

$$\langle f,g\rangle = \sum_{j=1}^{\infty} \langle f,e_j\rangle\langle e_j,g\rangle \quad \forall f,g \in \mathcal{H}.$$

Then $\langle h,g\rangle = \langle \sum\langle f,e_j\rangle e_j, g\rangle = \sum\langle f,e_j\rangle\langle e_j,g\rangle = \langle f,g\rangle$

This shows that $h=f$, hence $f = \sum\langle f,e_j\rangle e_j$

(⇐) $f = \sum\langle f,e_j\rangle e_j$ means that $\langle f,g\rangle = \sum\langle f,e_j\rangle\langle e_j,g\rangle \quad \forall f,g\in\mathcal{H}$

If $f=g$, then $\|f\|^2 = \sum|\langle f,e_j\rangle|^2$

1(b) Let $\mathcal{D}\subset\mathcal{H}$ be a dense subspace. For $f\in\mathcal{H}$, let $(f_n)\subset\mathcal{D}$ be s.t.

$$\lim_n \|f_n - f\| = 0$$

Given $N\in\mathbb{N}$,

$$\sum_{j=1}^{N} |\langle f,e_j\rangle|^2 = \cancel{\lim_{n\to\infty}|\langle f_n,e_j\rangle|^2} = \lim_{n\to\infty} \sum_{j=1}^{N} |\langle f_n,e_j\rangle|^2$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{\infty} |\langle f_n,e_j\rangle|^2 = \lim_{n\to\infty} \|f_n\|^2 = \|f\|^2$$

Since N is arbitrary, then $\sum_{j=1}^{\infty} |\langle f,e_j\rangle|^2 \le \|f\|^2$ (*)

Next, choose $\varepsilon>0$ and $g\in\mathcal{D}$ s.t. $\|f-g\|<\varepsilon$. Now

$$\|f\| - 2\varepsilon \le \|g\| - \varepsilon \le \|g\| - \|g-f\| = \Big(\sum_{j=1}^{\infty} |\langle g,e_j\rangle|^2\Big)^{1/2} - \|g-f\|$$

$$\le \Big(\sum_{j=1}^{\infty} |\langle g,e_j\rangle|^2\Big)^{1/2} - \Big(\sum_{j=1}^{\infty} |\langle g-f,e_j\rangle|^2\Big)^{1/2} \le \Big(\sum |\langle f,e_j\rangle|^2\Big)^{1/2}$$

by (*).

by Minkowski's ineq.

(2) Let (e_j) be a P.F. with $\|e_j\|=1 \quad \forall j$

$$1 = \|e_j\|^2 = \sum_{l=1}^{\infty} |\langle e_j, e_l\rangle|^2 = \|e_j\|^4 + \sum_{l\neq j} |\langle e_j, e_l\rangle|^2 = 1 + \sum_{l \neq j} |\langle e_j, e_l\rangle|^2$$

Thus $\langle e_j, e_l\rangle = 0$ if $j \neq l$.

(3) Since $\{\psi_{j,k}\}$ is an ON basis, then

$$\sum_{j\in\mathbb{Z}} |\hat\psi(2^j\xi)|^2 = 1 \quad \text{for a.e. } \xi$$

Arguing by contradiction, suppose that $\hat\psi(0) = \delta \neq 0$.

then $\sum_{j\in\mathbb{Z}} |\hat\psi(2^j 0)|^2 = \lim_{N\to\infty} \sum_{j=-N}^{N} |\hat\psi(2^j 0)|$ diverges.

This is a contradiction. It must be $\hat\psi(0) = 0$

Alternatively,

$$\int_1^2 \frac{1}{\xi} \sum_{j\in\mathbb{Z}} |\hat\psi(2^j\xi)|\, d\xi = \int_0^\infty \frac{|\hat\psi(u)|^2}{u}\, du = \int_1^2 \frac{d\xi}{\xi} = \ln 2$$

$u = 2^j \xi$

Arguing by contradiction, assume $\hat\psi(0) = \delta \neq 0$. Let $\varepsilon > 0$ be s.t. $|\hat\psi(\xi)| > \frac{\delta}{2}$ for any $\xi \in (0, \varepsilon)$. Then

$$\int_0^\infty \frac{|\hat\psi(\xi)|^2}{\xi}\, d\xi \geq \int_0^\varepsilon \frac{|\delta|^2}{4\xi}\, d\xi = \infty. \Rightarrow \text{CONTRADICTION}$$

(4) $\hat\psi = \chi_{[-1,1]\setminus[-1/2,1/2]}$. $\hat\psi_{j,k} = \chi_{I_{j}}\, e^{-2\pi i 2^{-j}k}$ where ~~$I_{j,k}$~~ $I_j = [2^j, 2^j) \setminus [-2^{j-1}, 2^{j-1}]$

the intervals I_j are pairwise disjoint and satisfy $\bigcup I_j = \mathbb{R}$

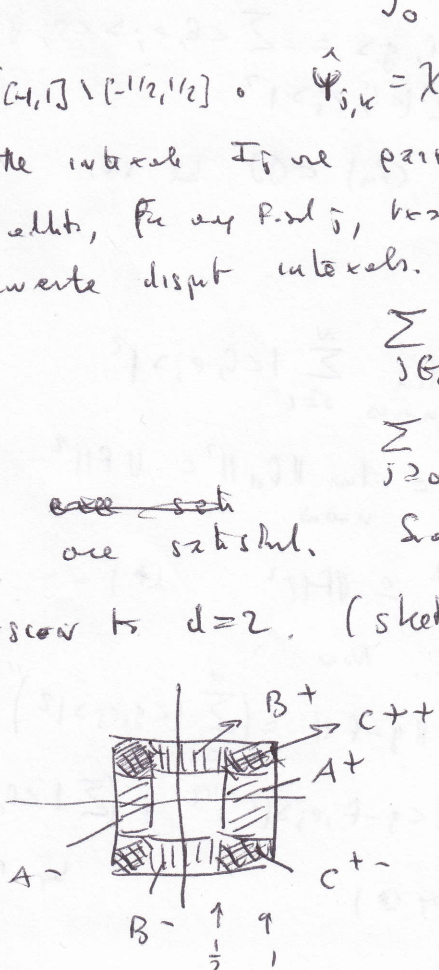
In addition, for any fixed j, translation by all integers of the INTERVALS I_j generate disjoint intervals. Hence the characteristic equations

$$\sum_{j\in\mathbb{Z}} |\hat\psi(2^j\xi)|^2 = 1 \quad \text{a.e.}$$

$$\sum_{j\geq 0} \hat\psi(2^j\xi)\, \overline{\hat\psi(2^j(\xi+k))} = 0.$$

~~are satis~~ are satisfied. Since $\|\psi_{j,k}\| = 1 \ \forall j,k$, the PF is an ONB.

EXTENSION to $d=2$. (sketch) $\quad \Psi = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})$



Need 3 wavelets

b/c WANT to make system O.N. → NORMALITY ~~Everything~~

$\hat\psi^{(1)} = \chi_{A^+ \cup A^-}$

$\hat\psi^{(2)} = \chi_{B^+ \cup B^-}$

$\hat\psi^{(3)} = \chi_{C^{++} \cup C^{+-} \cup C^{-+} \cup C^{--}}$