Math 6397 – Fall 2017 Name:

HW #4

(1) Let E be a measurable subset of $\mathbb{R}$ and $a > 1$ be fixed. We say that E tiles by translation if $\cup_{k\in\mathbb{Z}}(E+k) = \mathbb{R}$ up to a set of measure 0 and the overlaps $(E+j)\cap(E+k)$ have measure 0 if $j \neq k$. Similarly, we say that E tiles by dilation if $\cup_{n\in\mathbb{Z}}(a^n E) = \mathbb{R}$ up to a set of measure 0 and the overlaps $(a^n E)\cap(a^m E)$ have measure 0 if $n \neq m$.

(a) Prove that if E tiles by translation then $\{e^{2\pi ikx} : k \in \mathbb{Z}\}$ is an ONB of $L^2(E)$.

(b) Show that if E tiles both by translation and by dilation and $\hat{\psi} = \chi_E$, then $\{D_a^n T_k \psi : n, k \in \mathbb{Z}\}$ is an ONB of $L^2(\mathbb{R})$

(c) Suppose that $\{D_a^n T_k \psi : n, k \in \mathbb{Z}\}$, where $\hat{\psi} = \chi_E$, is an ONB of $L^2(\mathbb{R})$. Show that E tiles both by translation and by dilation.

(2) Suppose that $\phi \in L^2(\mathbb{R})$ is refinable with refinement coefficients $(c_k) \subset \ell^1$. Show that $T_m\phi$ is refinable and the corresponding refinement coefficients are (c_{k+m}).

(3) Fix $\phi \in L^2(\mathbb{R})$ and let $m(\xi) = \frac{1}{2}\sum_k c_k\, e^{-2\pi ik\xi}$ with $(c_k) \in \ell^2 \cap \ell^1$. Suppose that

$$\hat{\phi}(\xi) = m(\xi/2)\,\hat{\phi}(\xi/2) \quad a.e.$$

Prove that ϕ is refinable (Note: use fact that $(c_k) \in \ell^1$).

(4) Show that if $\sum_k |k^j c_k| < \infty$, then $m(\xi) = \frac{1}{2}\sum_k c_k\, e^{-2\pi ik\xi}$ is j times differentiable and

$$m^{(j)}(\xi) = \frac{(-2\pi i)^j}{2}\sum_k k^j c_k\, e^{-2\pi ik\xi}.$$

Derive that

$$m^{(j)}(0) = 0 \Leftrightarrow \sum_k (-1)^k k^j c_k = 0.$$

(5) Suppose that $\phi \in L^2(\mathbb{R})$ is refinable with refinement coefficients $(c_k) \subset \ell^2$.

(a) Prove that the series

$$\psi(x) = \sum_k (-1)^k \overline{c_{1-k}}\,\phi(2x-k)$$

converges unconditionally in $L^2(\mathbb{R})$.

(b) Prove that

$$\hat{\psi}(\xi) = m_1(\xi/2)\,\hat{\phi}(\xi/2), \quad \text{where} \quad m_1(\xi) = e^{-2\pi i\xi}\,\overline{m_0(\xi+1/2)}.$$