Math 4377/6308 Advanced Linear Algebra

1.2 Vector Spaces

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Vector Spaces: Introduction

Properties of $\mathbb{R}^n$

Many concepts concerning vectors in $\mathbb{R}^n$ can be extended to other mathematical systems.

- Parallelogram law for vector addition.
- Reading: §1.1.

Vector Spaces: Introduction (cont.)

We can think of a **vector space** in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called **vectors**.

Field

Let F be a **field**, whose elements are referred to as **scalars**.

- $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (rational numbers), etc.
- Reading: Appendix C.

Vector Spaces: Definition

Vector Space

A **vector space** over F is a nonempty set V, whose elements are referred to as **vectors**, together with two operations.

- The first operation, called **addition** and denoted by $+$, assigns to each pair $(\mathbf{u}, \mathbf{v})$ of vectors in V a vector $\mathbf{u} + \mathbf{v}$ in V (Axiom 1).
- The second operation, called **scalar multiplication** and denoted by juxtaposition, assigns to each pair $(a, \mathbf{u}) \in F \times V$ a vector $a\mathbf{u}$ in V (Axiom 6).

Furthermore, the following properties must be satisfied:

(VS 1) (**Commutativity of addition**) (Axiom 2) For all vectors $\mathbf{u}, \mathbf{v} \in V$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Vector Spaces: Definition (cont.)

Vector Space (cont.)

(VS 2) (**Associativity of addition**) (Axiom 3) For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$$

(VS 3) (**Existence of a zero**) (Axiom 4) There is a vector (called the zero vector) $\mathbf{0}$ in V such that

$$\mathbf{u}+\mathbf{0} = \mathbf{u}.$$

for all vectors $\mathbf{u} \in V$.

(VS 4) (**Existence of additive inverses**) (Axiom 5) For each vector $\mathbf{u}$ in V, there is a vector in V (called the additive inverse of $\mathbf{u}$), denoted by $-\mathbf{u}$, satisfying

$$\mathbf{u}+(-\mathbf{u}) = \mathbf{0}.$$

Vector Spaces: Definition (cont.)

Vector Space (cont.)

(VS 5-8) (**Properties of scalar multiplication**) (Axioms 7-10) For all scalars $a, b \in F$ and for all vectors $\mathbf{u}, \mathbf{v} \in V$,

$$\begin{aligned} 1\mathbf{u} &= \mathbf{u}. \\ (ab)\mathbf{u} &= a(b\mathbf{u}). \\ a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v}. \\ (a + b)\mathbf{u} &= a\mathbf{u} + b\mathbf{u}. \end{aligned}$$

A vector space over a field F is sometimes called an F-**space**. A vector space over the real field is called a **real vector space** and a vector space over the complex field is called a **complex vector space**.

Vector Spaces: Row and Column Vectors

Example

The set F^n of all ordered n-tuples whose components lie in a field F, is a vector space over F, with addition and scalar multiplication defined componentwise:

$$(a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (a_1 + b_1, \cdots, a_n + b_n)$$

and

$$c(a_1, \cdots, a_n) = (ca_1, \cdots, ca_n)$$

When convenient, we will also write the elements of F^n in column form

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Vector Spaces: 2×2 Matrices

Example

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\ b,\ c,\ d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} & \\ & \end{bmatrix}$.

Vector Spaces: $m \times n$ Matrices

Example

The set $\mathcal{M}_{m,n}(F)$ of all $m \times n$ matrices with entries in a field F of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

with $a_{ij} \in F$ for $1 \leq i \leq m$, $1 \leq j \leq n$, is a vector space over F, under the operations of matrix addition and scalar multiplication:

$$\begin{aligned} (A+B)_{ij} =& A_{ij} + B_{ij}, \\ (cA)_{ij} =& cA_{ij}, \end{aligned}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$.

Vector Spaces: Sequences

Example

Many sequence spaces are vector spaces. The set $\mathrm{Seq}(F)$ of all infinite sequences with members from a field F is a vector space under the componentwise operations

$$\{s_n\} + \{t_n\} = \{s_n + t_n\}$$

and

$$a\{s_n\} = \{as_n\}$$

Example (c_0)

In a similar way, the set c_0 of all sequences of complex numbers that converge to 0 is a vector space.

Example (l^∞)

The set l^∞ of all bounded complex sequences is a vector space.

Vector Spaces: Sequences (cont.)

Example (l^p)

If $1 \leq p < \infty$, then the set l^p of all complex sequences $\{s_n\}$ for which

$$\sum_{n=1}^{\infty} |s_n|^p < \infty$$

is a vector space under componentwise operations. To see that addition is a binary operation on l^p, one verifies Minkowski's inequality

$$\left(\sum_{n=1}^{\infty} |s_n + t_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |s_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |t_n|^p\right)^{1/p}$$

which we will not do here.

Vector Spaces: Functions

Example

Let $\mathcal{F}(S, F)$ denote the set of all functions from a nonempty set S to a field F. This is a vector space over F, under the operations of ordinary addition and scalar multiplication of functions:

$$(f + g)(s) = f(s) + g(s),$$

and

$$(af)(s) = a[f(s)],$$

for each $s \in S$.

Vector Spaces: Polynomials

Example

Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n \geq 0.$$

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable. The set $\mathbf{P}_n$ is a vector space.

We will just verify 3 out of the 10 axioms here.

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$ (set higher coefficients to zero if different degrees). Let c be a scalar.

Vector Spaces: Polynomials (cont.)

Axiom 1:

The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows:
$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$$

$$= (\_\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_\_)\, t + \cdots + (\_\_\_\_\_\_\_\_\_)\, t^n$$

which is also a ____________________ of degree at most ________. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$.

Vector Spaces: Polynomials (cont.)

Axiom 4:

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$

(zero vector in $\mathbf{P}_n$)

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n$$

$$= a_0 + a_1 t + \cdots + a_n t^n = \mathbf{p}(t)$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$

Vector Spaces: Polynomials (cont.)

Axiom 6:

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_)\,t + \cdots + (\_\_\_\_\_\_\_\_)\,t^n$$

which is in $\mathbf{P}_n$.

The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

Vector Spaces: True or False

1. Every vector space contains a zero vector.
2. A vector space may have more than one zero vector.
3. In any vector space, $ax = bx$ implies that $a = b$.
4. In any vector space, $ax = ay$ implies that $x = y$.
5. A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$.
6. An $m \times n$ matrix has m columns and n rows.
7. In $P(F)$, only polynomials of the same degree may be added.
8. In f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
9. If f is a polynomial of degree n and c is nonzero scalar, then cf is a polynomial of degree n.
10. A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero.
11. Two functions in $F(S, F)$ are equal if and only if they have the same value at each element of S.

Vector Spaces: Properties

Theorem (1.1 Cancellation Law for Vector Addition)

If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *are vectors in a vector space* V *such that* $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$, *then* $\mathbf{x} = \mathbf{y}$.

Vector Spaces: Properties (cont.)

Corollary 1 (Uniqueness of the Zero Vector)

The vector $\mathbf{0}$ described in (VS 3) is unique (the zero vector).

Vector Spaces: Properties (cont.)

Corollary 2 (Uniqueness of the Additive Inverse)

The vector $-\mathbf{u}$ described in (VS 4) is unique (the additive inverse).

Vector Spaces: Properties (cont.)

Theorem (1.2)

In any vector space V, the following statements are true:

(a) $0\mathbf{x} = \mathbf{0}$ *for each* $\mathbf{x} \in V$.

(b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ *for each* $a \in F$ *and* $\mathbf{x} \in V$

(c) $a\mathbf{0} = \mathbf{0}$ *for each* $a \in F$