LECTURE 5

Review of Set Theory

1. Sets

For our purposes, a **set** is any collection of objects. One can specify a set either in prose, e.g.,

The set $\mathbb{Z}$ of integers.

Or one can specify the elements of a set by listing its elements, e.g,,

$$\{1,3,7\}$$

$$\{0,2,4,6,8,\cdots\}$$

$$\{x \mid x \in \mathbb{Z} \text{ and } x > 3\}$$

We note that what is meant in the second line is "the set of all positive even integers" and that what is meant in the third line is "the set of all elements x such that x is an integer and x is greater than 3".

1.1. The Empty Set.

The *empty set*, which is denoted by $\emptyset$, or $\{\ \}$; is the set with no elements. While seemingly non-sensical, the empty set is very nice concept to have around. Its utility in proofs is something akin to the use of 0 in common arithmetic.

1.2. Subsets.

A set B is said to be a subset of a set C (written $B \subseteq C$) if every element of B is also an element of C. In mathematical short hand, this definition is written

$$B \subseteq C \Leftrightarrow \forall\, x \in B, \quad x \in C \quad .$$

Note that the definition of a subset B of C allows for the possibility that $B = C$. In fact,

$$C \subseteq C\ .$$

That is to say, **every subset is a subset of itself**..

Another consequence of this definition is the following special property of the empty set.

PROPOSITION 5.1. *The empty set is a subset of every set.*

Proof. $\emptyset$, like any other set, is a subset of itself. Let C be an arbitrary set and let x be an arbitrary element of C. Since $\emptyset$ is the empty set, the statement

$$x \in \emptyset$$

is false. This means that the proposition

$$x \in \emptyset \Rightarrow x \in C$$

is always true (by a quirk of logic; if the premise of a conditional statement is always false, then the conditional statement itself is always true)[1]. But this is just the definition $\emptyset$ as a subset of C. ∎

[1] Note, however that the conditional statement,

$$x \in \emptyset \quad \Rightarrow \quad x \notin C$$

We also remark that the subset relation is *transitive*; i.e.,

$$B \subseteq C \textbf{ and } C \subseteq D \quad \Rightarrow \quad B \subseteq D \quad .$$

and that

$$B = C \quad \Leftrightarrow \quad B \subseteq C \textbf{ and } C \subseteq B \quad .$$

2. Operations on Sets

If A and B are sets, then the **relative complement** of A in B, denoted

$$B - A$$

is the set of elements of B that are not elements of A:

$$B - A = \{x \in B \mid x \notin A\} \quad .$$

The **intersection**, $A \cap B$, of sets A and B is the set consisting of all elements of A which are also elements of B:

$$A \cap B = \{x \in A \mid x \in B\},$$

or equivalently,

$$A \cap B = \{x \in B \mid x \in A\} \quad .$$

If A and B are two sets with no common elements

$$A \cap B = \emptyset$$

and we say that A and B are **disjoint**.

The **union**, $A \cup B$, of two sets A and B is the set consisting of all elements that are in A and/or B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \quad .$$

The concepts of intersection and union extend readily to large (possibly infinite) collections of sets. At such times, we will use the following notation.

Suppose that I is a non-empty set (called an **index set**) and that for each $i \in I$, we are given a set A_i. Then the intersection of this family of sets, denoted $\cap_{i \in I} A_i$ is the set

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \ , \ \forall \ i \in I\} \quad ,$$

and the union of this family of sets, denoted $\cup_{i \in I} A_i$, is the set

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_j \text{ for some } j \in I\} \quad .$$

EXAMPLE 5.2. Let $I = \mathbb{N}$, the set of non-negative integers, and set

$$A_n = \{x \in \mathbb{Z} \mid |x| \leq n\}$$

Then

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}$$

and

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{Z} \quad .$$

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is also true. Note also, that the definition of **subset** does not specifically require that $x \in C$. It only requires the truth of the conditional statement

$$x \in \emptyset \quad \Rightarrow \quad x \in C$$

Finally, we define the **Cartesian product**, $A \times B$, of two sets A and B as the set consisting of all ordered pairs (x, y) with $x \in A$ and $y \in B$.

Homework:

1. Prove the following identities:

$$B \cap (C \cup D) = (B \cap C) \cup (B \cap D) \tag{5.1}$$

$$B \cup (C \cap D) = (B \cup C) \cap (B \cup D) \tag{5.2}$$

$$C = (C - A) \cup (C \cap A) \tag{5.3}$$

2. Do Problems 2, 3, 4, and 6 on pages 492-493.