

CHAPTER 13

SECTION 13.1

3. $\mathbf{f}'(t) = -\frac{1}{2\sqrt{1-t}}\mathbf{i} + \frac{1}{2\sqrt{1+t}}\mathbf{j} + \frac{1}{(1-t)^2}\mathbf{k}$ 4. $\mathbf{f}'(t) = e^t\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$
7. $\mathbf{f}'(t) = \frac{-1}{1-t}\mathbf{i} - \sin t\mathbf{j} + 2t\mathbf{k}$ 8. $\mathbf{f}'(t) = -\frac{2}{(t-1)^2}\mathbf{i} + e^{2t}(1+2t)\mathbf{j} + \sec t \tan t\mathbf{k}$
7. $\mathbf{f}'(t) = \frac{-1}{1-t}\mathbf{i} - \sin t\mathbf{j} + 2t\mathbf{k}$
9. $\mathbf{f}'(t) = 4\mathbf{i} + 6t^2\mathbf{j} + (2t+2)\mathbf{k}; \quad \mathbf{f}''(t) = 12t\mathbf{j} + 2\mathbf{k}$
10. $\mathbf{f}'(t) = (\sin t + t \cos t)\mathbf{i} + (\cos t - t \sin t)\mathbf{k}$
 $\mathbf{f}''(t) = (2 \cos t - t \sin t)\mathbf{i} + (-2 \sin t - t \cos t)\mathbf{k}$
16. $\int_0^\pi \mathbf{r}(t) dt = \left[-\cos t\mathbf{i} + \sin t\mathbf{j} + \frac{t^2}{2}\mathbf{k} \right]_0^\pi = 2\mathbf{i} + \frac{1}{2}\pi^2\mathbf{k}$
19. $\int_0^1 \left(\frac{1}{1+t^2}\mathbf{i} + \sec^2 t\mathbf{j} \right) dt = [\tan^{-1} t\mathbf{i} + \tan t\mathbf{j}]_0^1 = \frac{\pi}{4}\mathbf{i} + \tan(1)\mathbf{j}$
20. $\int_1^3 \mathbf{F}(t) dt = \left[\ln t\mathbf{i} + \frac{1}{2}(\ln t)^2\mathbf{j} - \frac{1}{2}e^{-2t}\mathbf{k} \right]_1^3 = \ln 3\mathbf{i} + \frac{1}{2}(\ln 3)^2\mathbf{j} + \frac{1}{2}(e^{-2} - e^{-6})\mathbf{k}$
21. $\lim_{t \rightarrow 0} \mathbf{f}(t) = \left(\lim_{t \rightarrow 0} \frac{\sin t}{2t} \right) \mathbf{i} + \left(\lim_{t \rightarrow 0} e^{2t} \right) \mathbf{j} + \left(\lim_{t \rightarrow 0} \frac{t^2}{e^t} \right) \mathbf{k} = \frac{1}{2}\mathbf{i} + \mathbf{j}$
22. Does not exist (because of $\frac{t}{|t|}\mathbf{k}$)
39. (a) $\mathbf{f}(t) = 3 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$ (b) $\mathbf{f}(t) = 3 \cos t\mathbf{i} - 2 \sin t\mathbf{j}$
40. (a) $\mathbf{f}(t) = (1 + \cos t)\mathbf{i} + \sin t\mathbf{j}$ (b) $\mathbf{f}(t) = (1 + \cos t)\mathbf{i} - \sin t\mathbf{j}$
43. $\mathbf{f}(t) = (1 + 2t)\mathbf{i} + (4 + 5t)\mathbf{j} + (-2 + 8t)\mathbf{k}, \quad 0 \leq t \leq 1$
46. $\mathbf{f}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + (\sqrt{1+t^2} + 1)\mathbf{j} + (te^t - e^t + 4)\mathbf{k}$
49. $\mathbf{f}'(t) = \alpha \mathbf{f}(t) \implies \mathbf{f}(t) = e^{\alpha t} \mathbf{f}(0) = e^{\alpha t} \mathbf{c}$

Section 13.2

3. $\mathbf{f}'(t) = 2e^{2t}\mathbf{i} - \cos t\mathbf{j}, \quad \mathbf{f}''(t) = 4e^{2t}\mathbf{i} + \sin t\mathbf{j}$
4. $\mathbf{f}(t) = 2t^2\mathbf{i} \implies \mathbf{f}'(t) = 4t\mathbf{i}, \quad \mathbf{f}''(t) = 4\mathbf{i}$

$$\begin{aligned}
7. \quad \mathbf{f}'(t) &= \left[(e^t \mathbf{i} + t \mathbf{k}) \times \frac{d}{dt} (t \mathbf{j} + e^{-t} \mathbf{k}) \right] + \left[\frac{d}{dt} (e^t \mathbf{i} + t \mathbf{k}) \times (t \mathbf{j} + e^{-t} \mathbf{k}) \right] \\
&= [(e^t \mathbf{i} + t \mathbf{k}) \times (\mathbf{j} - e^{-t} \mathbf{k})] + [(e^t \mathbf{i} + \mathbf{k}) \times (t \mathbf{j} + e^{-t} \mathbf{k})] \\
&= (-t \mathbf{i} + \mathbf{j} + e^t \mathbf{k}) + (-t \mathbf{i} - \mathbf{j} + t e^t \mathbf{k}) \\
&= -2t \mathbf{i} + e^t (t + 1) \mathbf{k}
\end{aligned}$$

$$\mathbf{f}''(t) = -2 \mathbf{i} + e^t (t + 2) \mathbf{k}$$

$$8. \quad \mathbf{f}'(t) = (\mathbf{i} - 2t \mathbf{j}) \times (\mathbf{i} + t^3 \mathbf{j} + 5t \mathbf{k}) + (t \mathbf{i} - t^2 \mathbf{j} + \mathbf{k}) \times (3t^2 \mathbf{j} + 5 \mathbf{k})$$

$$\mathbf{f}''(t) = -2 \mathbf{j} \times (\mathbf{i} + t^3 \mathbf{j} + 5 \mathbf{k}) + 2(\mathbf{i} - 2t \mathbf{j}) \times (3t^2 \mathbf{j} + 5 \mathbf{k}) + (t \mathbf{i} - t^2 \mathbf{j} + \mathbf{k}) \times 6t \mathbf{j}$$

$$\begin{aligned}
14. \quad \frac{d^2}{dt^2} [e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j}] &= \frac{d}{dt} [e^t (\cos t - \sin t) \mathbf{j} + e^t (\sin t + \cos t) \mathbf{j}] \\
&= -2e^t \sin t \mathbf{i} + 2e^t \cos t \mathbf{j}
\end{aligned}$$

$$15. \quad (e^t \mathbf{i} + e^{-t} \mathbf{j}) \cdot (e^t \mathbf{i} - e^{-t} \mathbf{j}) = e^{2t} - e^{-2t}; \quad \text{therefore}$$

$$\frac{d^2}{dt^2} [(e^t \mathbf{i} + e^{-t} \mathbf{j}) \cdot (e^t \mathbf{i} - e^{-t} \mathbf{j})] = \frac{d^2}{dt^2} [e^{2t} - e^{-2t}] = \frac{d}{dt} [2e^{2t} + 2e^{-2t}] = 4e^{2t} - 4e^{-2t}$$

$$16. \quad \left(\frac{1}{t} \mathbf{i} + \mathbf{j} - 2t \mathbf{k} \right) \times \left(\frac{1}{t} \mathbf{i} + t^2 \mathbf{j} - t \mathbf{k} \right) + [\ln t \mathbf{i} + t \mathbf{j} - (t^2 + 1) \mathbf{k}] \times \left(-\frac{1}{t^2} \mathbf{i} + 2t \mathbf{j} - \mathbf{k} \right)$$

$$21. \quad \mathbf{r}(t) = \mathbf{a} + t \mathbf{b}$$

$$22. \quad \mathbf{r}(t) = \mathbf{a} + t \mathbf{b} + \frac{1}{2} t^2 \mathbf{c}$$

$$24. \quad \mathbf{r}(t) = (1 + 2t - \frac{1}{4} \cos 2t) \mathbf{i} + (1 - \frac{1}{4} \sin 2t) \mathbf{j}$$

$$25. \quad \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j}, \quad \mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j}, \quad \mathbf{r}''(t) = -\sin t \mathbf{i} - \cos t \mathbf{j} = -\mathbf{r}(t).$$

Thus $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are parallel, and they always point in opposite directions.

$$27. \quad \mathbf{r}(t) \cdot \mathbf{r}'(t) = (\cos t \mathbf{i} + \sin t \mathbf{j}) \cdot (-\sin t \mathbf{i} + \cos t \mathbf{j}) = 0$$

$$\begin{aligned}
\mathbf{r}(t) \times \mathbf{r}'(t) &= (\cos t \mathbf{i} + \sin t \mathbf{j}) \times (-\sin t \mathbf{i} + \cos t \mathbf{j}) \\
&= \cos^2 t \mathbf{k} + \sin^2 t \mathbf{k} = (\cos^2 t + \sin^2 t) \mathbf{k} = \mathbf{k}
\end{aligned}$$

32. $\frac{d}{dt} (\mathbf{f} \times \mathbf{f}') = \mathbf{f}(t) \times \mathbf{f}''(t)$ by Exercise 29. If $\mathbf{f}(t)$ and $\mathbf{f}''(t)$ are parallel, their cross product is zero, so $\frac{d}{dt} (\mathbf{f} \times \mathbf{f}') = \mathbf{0}$, hence $\mathbf{f} \times \mathbf{f}'$ is constant.

$$\begin{aligned}
33. \quad \|\mathbf{r}(t)\| \text{ is constant} &\iff \|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t) \text{ is constant} \\
&\iff \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{r}(t)] = 2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 0 \text{ identically} \\
&\iff \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \text{ identically}
\end{aligned}$$

Section 13.3

1. $\mathbf{r}'(t) = -\pi \sin \pi t \mathbf{i} + \pi \cos \pi t \mathbf{j} + \mathbf{k}$, $\mathbf{r}'(2) = \pi \mathbf{j} + \mathbf{k}$
 $\mathbf{R}(u) = (\mathbf{i} + 2\mathbf{k}) + u(\pi \mathbf{j} + \mathbf{k})$
2. $\mathbf{r}'(t) = e^t \mathbf{i} - e^{-t} \mathbf{j} - \frac{1}{t} \mathbf{k}$, $\mathbf{r}'(1) = e \mathbf{i} - e^{-1} \mathbf{j} - \mathbf{k}$; $\mathbf{R}(u) = (e \mathbf{i} + e^{-1} \mathbf{j}) + u(e \mathbf{i} - e^{-1} \mathbf{j} - \mathbf{k})$
4. $\mathbf{r}'(0) = \mathbf{i}$; $\mathbf{R}(u) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + u \mathbf{i}$.
5. $\mathbf{r}'(t) = 4t \mathbf{i} - \mathbf{j} + 4t \mathbf{k}$, P is tip of $\mathbf{r}(1)$, $\mathbf{r}'(1) = 4 \mathbf{i} - \mathbf{j} + 4 \mathbf{k}$
 $\mathbf{R}(u) = (2 \mathbf{i} + 5 \mathbf{k}) + u(4 \mathbf{i} - \mathbf{j} + 4 \mathbf{k})$
9. The scalar components $x(t) = at$ and $y(t) = bt^2$ satisfy the equation
$$a^2 y(t) = a^2 (bt^2) = b(a^2 t^2) = b[x(t)]^2$$
and generate the parabola $a^2 y = bx^2$.

11. $\mathbf{r}(t) = t \mathbf{i} + (1 + t^2) \mathbf{j}$, $\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j}$

(a) $\mathbf{r}(t) \perp \mathbf{r}'(t) \implies \mathbf{r}(t) \cdot \mathbf{r}'(t) = [t \mathbf{i} + (1 + t^2) \mathbf{j}] \cdot (\mathbf{i} + 2t \mathbf{j})$
 $= t(2t^2 + 3) = 0 \implies t = 0$

$\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular at $(0, 1)$.

(b) and (c) $\mathbf{r}(t) = \alpha \mathbf{r}'(t)$ with $\alpha \neq 0 \implies t = \alpha$ and $1 + t^2 = 2t\alpha \implies t = \pm 1$.

If $\alpha > 0$, then $t = 1$. $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the same direction at $(1, 2)$.

If $\alpha < 0$, then $t = -1$. $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions at $(-1, 2)$.

17. $\mathbf{r}_1(t) = \mathbf{r}_2(u)$ implies

$$\left\{ \begin{array}{l} e^t = u \\ 2 \sin \left(t + \frac{1}{2} \pi \right) = 2 \\ t^2 - 2 = u^2 - 3 \end{array} \right\} \text{ so that } t = 0, \quad u = 1.$$

The point of intersection is $(1, 2, -2)$.

$$\mathbf{r}'_1(t) = e^t \mathbf{i} + 2 \cos \left(t + \frac{\pi}{2} \right) \mathbf{j} + 2t \mathbf{k}, \quad \mathbf{r}'_1(0) = \mathbf{i}$$

$$\mathbf{r}'_2(u) = \mathbf{i} + 2u \mathbf{k}, \quad \mathbf{r}'_2(1) = \mathbf{i} + 2 \mathbf{k}$$

$$\cos \theta = \frac{\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(1)}{\|\mathbf{r}'_1(0)\| \|\mathbf{r}'_2(1)\|} = \frac{1}{5} \sqrt{5} \cong 0.447, \quad \theta \cong 1.11 \text{ radians}$$

18. (a) $\mathbf{R}(u) = \mathbf{r}(t_0) + u \mathbf{r}'(t_0) = (t_0 \mathbf{i} + f(t_0) \mathbf{j}) + u(\mathbf{i} + f'(t_0) \mathbf{j})$

$$\implies x(u) = t_0 + u, \quad y(u) = f(t_0) + u f'(t_0)$$

(b) From (a), we get $u = x(u) - t_0$ and $y(u) - f(t_0) = f'(t_0)u$. so

$$y - f(t_0) = f'(t_0)(x - t_0), \quad \text{as expected.}$$

19. (a) $\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}$ (b) $\mathbf{r}(t) = a \cos t \mathbf{i} - b \sin t \mathbf{j}$
(c) $\mathbf{r}(t) = a \cos 2t \mathbf{i} + b \sin 2t \mathbf{j}$ (d) $\mathbf{r}(t) = a \cos 3t \mathbf{i} - b \sin 3t \mathbf{j}$

27. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + t\mathbf{j}, \quad t \geq 1;$ or, $\mathbf{r}(t) = \sec^2 t \mathbf{i} + \tan t \mathbf{j}, \quad t \in [\frac{1}{4}\pi, \frac{1}{2}\pi)$

29. $\mathbf{r}(t) = \cos t \sin 3t \mathbf{i} + \sin t \sin 3t \mathbf{j}, \quad t \in [0, \pi]$

32. (a) $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} = \mathbf{r}(1)$

(b) $\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{1}{\sqrt{\pi^2 + 2}}(-\mathbf{i} - \mathbf{j} + \pi\mathbf{k}), \quad \mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{\pi^2 + 5}}(\mathbf{i} + 2\mathbf{j} - \pi\mathbf{k})$

35. $\mathbf{r}'(t) = 2\mathbf{j} + 2t\mathbf{k}, \quad \|\mathbf{r}'(t)\| = 2\sqrt{1 + t^2}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + t^2}}(\mathbf{j} + t\mathbf{k}),$$

$$\mathbf{T}'(t) = \frac{1}{(1 + t^2)^{3/2}}[-t\mathbf{j} + \mathbf{k}]$$

at $t = 1$: tip of $\mathbf{r} = (1, 2, 1), \quad \mathbf{T} = \mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k};$

$$\mathbf{T}'(1) = -\frac{1}{2\sqrt{2}}\mathbf{j} + \frac{1}{2\sqrt{2}}\mathbf{k}; \quad \|\mathbf{T}'(1)\| = \frac{1}{2}; \quad \mathbf{N} = \mathbf{N}(1) = \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

normal for osculating plane:

$$\mathbf{T} \times \mathbf{N} = \left(\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) \times \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{1}{2}\mathbf{i}$$

equation for osculating plane:

$$\frac{1}{2}(x - 1) + 0(y - 2) + 0(z - 1) = 0, \quad \text{which gives } x - 1 = 0$$

37. $\mathbf{r}'(t) = -2 \sin 2t \mathbf{i} + 2 \cos 2t \mathbf{j} + \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{5}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{5}}(-2 \sin 2t \mathbf{i} + 2 \cos 2t \mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = -\frac{4}{\sqrt{5}}(\cos 2t \mathbf{i} + \sin 2t \mathbf{j}), \quad \|\mathbf{T}'(t)\| = \frac{4}{\sqrt{5}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -(\cos 2t \mathbf{i} + \sin 2t \mathbf{j})$$

at $t = \pi/4$: tip of $\mathbf{r} = (0, 1, \pi/4), \quad \mathbf{T} = \frac{1}{\sqrt{5}}\sqrt{5}(-2\mathbf{i} + \mathbf{k}), \quad \mathbf{N} = -\mathbf{j}$

normal for osculating plane:

$$\mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}}\sqrt{5}(-2\mathbf{i} + \mathbf{k}) \times (-\mathbf{j}) = \frac{1}{\sqrt{5}}\sqrt{5}\mathbf{i} + \frac{2}{\sqrt{5}}\sqrt{5}\mathbf{k}$$

equation for osculating plane:

$$\frac{1}{5}\sqrt{5}(x-0) + \frac{2}{5}\sqrt{5}\left(z - \frac{\pi}{4}\right) = 0, \quad \text{which gives } x + 2z = \frac{\pi}{2}$$

$$\begin{aligned} 38. \quad \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}}{\sqrt{4t^2 + 5}} & \mathbf{T}(2) &= \frac{1}{\sqrt{21}}(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \\ \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{-2t\mathbf{i} - 4t\mathbf{j} + 5\mathbf{k}}{\sqrt{20t^2 + 25}} & \mathbf{N}(2) &= \frac{1}{\sqrt{105}}(-4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}) \\ \mathbf{T}(2) \times \mathbf{N}(2) &= \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j}) \end{aligned}$$

$$\text{Osculating plane at } (2, 4, 2): \quad 2(x-2) - (y-4) = 0 \implies 2x - y = 0$$

$$\begin{aligned} 41. \quad \mathbf{r}'(t) &= e^t[(\sin t + \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j} + \mathbf{k}], \quad \|\mathbf{r}'(t)\| = e^t\sqrt{3} \\ \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{3}}[(\sin t + \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j} + \mathbf{k}], \\ \mathbf{T}'(t) &= \frac{1}{\sqrt{3}}[(\cos t - \sin t)\mathbf{i} - (\sin t + \cos t)\mathbf{j}] \end{aligned}$$

$$\text{at } t = 0: \quad \text{tip of } \mathbf{r} = (0, 1, 1), \quad \mathbf{T} = \mathbf{T}(0) = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k});$$

$$\mathbf{T}'(0) = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j}); \quad \|\mathbf{T}'(0)\| = \frac{\sqrt{2}}{\sqrt{3}}; \quad \mathbf{N} = \mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$$

normal for osculating plane:

$$\mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

equation for osculating plane:

$$\frac{1}{\sqrt{6}}(x-0) + \frac{1}{\sqrt{6}}(y-1) - \frac{2}{\sqrt{6}}(z-1) = 0, \quad \text{which gives } x + y - 2z + 1 = 0$$

Section 13.4

$$1. \quad \mathbf{r}'(t) = \mathbf{i} + t^{1/2}\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{1+t} \qquad 2. \quad \mathbf{r}'(t) = (t^2-1)\mathbf{i} + 2t\mathbf{j}; \quad \|\mathbf{r}'(t)\| = t^2+1$$

$$L = \int_0^8 \sqrt{1+t} dt = \left[\frac{2}{3}(1+t)^{3/2} \right]_0^8 = \frac{52}{3} \qquad L = \int_0^2 (t^2+1) dt = \frac{14}{3}$$

$$3. \quad \mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{a^2 + b^2}$$

$$L = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2}$$

$$4. \quad \mathbf{r}'(t) = \mathbf{i} + \sqrt{2}t^{1/2}\mathbf{j} + t\mathbf{k}; \quad \|\mathbf{r}'(t)\| = t+1$$

$$L = \int_0^2 (t+1) dt = 4$$

$$8. \quad \mathbf{r}'(t) = \mathbf{i} + \left(\frac{1}{2}t^2 - \frac{1}{2}t^{-2}\right) \mathbf{k}; \quad \|\mathbf{r}'(t)\| = \frac{1}{2}t^2 + \frac{1}{2}t^{-2}$$

$$L = \int_1^3 \left(\frac{1}{2}t^2 + \frac{1}{2}t^{-2}\right) dt = \frac{14}{3}$$

$$13. \quad \mathbf{r}'(t) = \frac{1}{t} \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t^2} + 4 + 4t^2}$$

$$L = \int_1^e \sqrt{\frac{1}{t^2} + 4 + 4t^2} dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = [\ln |t| + t^2]_1^e = e^2$$

$$15. \quad \mathbf{r}'(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j} + \sqrt{3}t \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t + 3t^2} = \sqrt{4t^2} = 2t$$

$$L = \int_0^{2\pi} 2t dt = [t^2]_0^{2\pi} = 4\pi^2$$

$$20. \quad L_1 = \int_1^e \|\mathbf{r}'(t)\| dt = \int_1^e \sqrt{2 + \frac{2}{t^2}} dt; \quad L_2 = \int_0^1 \sqrt{1 + e^{2x}} dx$$

Setting $t = e^x$, we get $L_1 = \sqrt{2}L_2$

$$23. \quad (a) \quad s = \int_0^t \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2} dt = \int_0^t 5 dt = 5t$$

$$(b) \quad t = \frac{s}{5}; \quad \mathbf{R}(s) = 3 \cos\left(\frac{s}{5}\right) \mathbf{i} + 3 \sin\left(\frac{s}{5}\right) \mathbf{j} + \frac{4s}{5} \mathbf{k}$$

$$(c) \quad \mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 4t \mathbf{k} = 3 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} \implies t = 0$$

From part (a), the arc length $s = 5t$ and $5t = 5\pi \implies t = \pi$; $\mathbf{r}(\pi) = -3 \mathbf{i} + 0 \mathbf{j} + 4\pi \mathbf{k}$

$$\implies Q(-3, 0, 4\pi).$$

$$(d) \quad \mathbf{R}'(s) = -\frac{3}{5} \sin\left(\frac{s}{5}\right) \mathbf{i} + \frac{3}{5} \cos\left(\frac{s}{5}\right) \mathbf{j} + \frac{4}{5} \mathbf{k};$$

$$\|\mathbf{R}'(s)\| = \sqrt{\left[-\frac{3}{5} \sin\left(\frac{s}{5}\right)\right]^2 + \left[\frac{3}{5} \cos\left(\frac{s}{5}\right)\right]^2 + \left[\frac{4}{5}\right]^2} = 1$$

Section 13.5

$$3. \quad \mathbf{r}(t) = at\mathbf{i} + b \sin at \mathbf{j}, \quad \mathbf{r}'(t) = a\mathbf{i} + ab \cos at \mathbf{j}$$

$$\mathbf{r}''(t) = -a^2b \sin at \mathbf{j}, \quad \|\mathbf{r}''(t)\| = a^2|b \sin at| = a^2|y(t)|$$

4. $\mathbf{r}'(t) = 2t\mathbf{j} + 2(t-1)\mathbf{k}$; speed is minimum when $\|\mathbf{r}'(t)\|^2$ is minimum

$$4t^2 + 4(t-1)^2 = 8t^2 - 8t + 4 \text{ is minimum at } t = \frac{1}{2}$$

10. (a) $x(t) = 2\cos 2t = 4\cos^2 t - 2$ and $y(t) = 3\cos t \implies x = \frac{4}{9}y^2 - 2$. Since

$$-2 \leq x(t) \leq 2, \quad -3 \leq y(t) \leq 3,$$

the path consists only of the bounded arc

$$x = \frac{4}{9}y^2 - 2, \quad -3 \leq y \leq 3.$$

The motion traces out this arc twice on every t -interval of length 2π .

- (b) included in part (d)

13. $\kappa = \frac{e^{-x}}{(1 + e^{-2x})^{3/2}}$

17. $\kappa = \frac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = |\cos x|$

18. $y' = \sec^2 x, \quad y'' = 2\sec^2 x \tan x; \quad \kappa = \frac{|2\sec^2 x \tan x|}{(1 + \sec^4 x)^{3/2}}$

21. $\kappa = \frac{|x|}{(1 + x^4/4)^{3/2}}; \quad \text{at } \left(2, \frac{4}{3}\right), \quad \rho = \frac{5}{2}\sqrt{5}$

24. $y' = 4\cos 2x, \quad y'' = -8\sin 2x; \quad \text{at } x = \frac{\pi}{4}, \quad y' = 0, \quad y'' = -8 \implies \kappa = \frac{8}{1} \implies \rho = \frac{1}{8}$

27. $\kappa(x) = \frac{|-1/x^2|}{(1 + 1/x^2)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}, \quad x > 0$

$$\kappa'(x) = \frac{(1 - 2x^2)}{(x^2 + 1)^{5/2}}, \quad \kappa'(x) = 0 \implies x = \frac{1}{2}\sqrt{2}$$

Since κ increases on $(0, \frac{1}{2}\sqrt{2}]$ and decreases on $[\frac{1}{2}\sqrt{2}, \infty)$, κ is maximal at $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\ln \frac{1}{2})$.

28. $y' = 3 - 3x^2, \quad y'' = -6x$. local max at $x = 1: \quad y' = 0, \quad y'' = -6, \quad \kappa = \frac{6}{(1+0)^{3/2}} = 6$

31. $x(t) = 2t, \quad x'(t) = 2, \quad x''(t) = 0; \quad y(t) = t^3, \quad y'(t) = 3t^2, \quad y''(t) = 6t;$

$$\kappa = \frac{12|t|}{(4 + 9t^4)^{3/2}}$$

32. $x' = 2t, \quad x'' = 2, \quad y' = 3t^2, \quad y'' = 6t; \quad \kappa = \frac{|12t^2 - 6t^2|}{(4t^2 + 9t^4)^{3/2}} = \frac{6}{|t|(4 + 9t^2)^{3/2}}$

35. $x(t) = t \cos t, \quad x'(t) = \cos t - t \sin t, \quad x''(t) = -2 \sin t - t \cos t$

$y(t) = t \sin t, \quad y'(t) = \sin t + t \cos t, \quad y''(t) = 2 \cos t - t \sin t$

$$\kappa = \frac{|(\cos t - t \sin t)(2 \cos t - t \sin t) - (\sin t + t \cos t)(-2 \sin t - t \cos t)|}{[(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2]^{3/2}} = \frac{2 + t^2}{[1 + t^2]^{3/2}}$$

40. $x' = r(1 - \cos t), \quad x'' = r \sin t, \quad y' = r \sin t, \quad y'' = r \cos t$

Highest point when $t = \pi \implies x' = 2r, \quad x'' = 0, \quad y' = 0, \quad y'' = -r$

$$\kappa = \frac{2r^2}{(4r^2)^{3/2}} = \frac{1}{4r}$$

41. $\mathbf{r}'(t) = e^t(\cos t - \sin t) \mathbf{i} + e^t(\sin t + \cos t) \mathbf{j} + e^t \mathbf{k}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{3} e^t, \quad \frac{d^2s}{dt^2} = \sqrt{3} e^t$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{3}} [(\cos t - \sin t) \mathbf{i} + (\sin t + \cos t) \mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}} [(-\sin t - \cos t) \mathbf{i} + (\cos t - \sin t) \mathbf{j}]$$

Then,

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{\sqrt{2/3}}{\sqrt{3} e^t} = \frac{1}{3} \sqrt{2} e^{-t},$$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = \sqrt{3} e^t, \quad \mathbf{a}_N = \kappa \left(\frac{ds}{dt}\right)^2 = \sqrt{2} e^t.$$

43. $\mathbf{r}'(t) = -2 \sin 2t \mathbf{i} + 2 \cos 2t \mathbf{j}; \quad \frac{ds}{dt} = \|\mathbf{r}'(t)\| = 2, \quad \frac{d^2s}{dt^2} = 0$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin 2t \mathbf{i} + \cos 2t \mathbf{j}$$

$$\mathbf{T}'(t) = -2(\cos 2t \mathbf{i} + \sin 2t \mathbf{j})$$

Then,

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{2}{2} = 1,$$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = 0, \quad \mathbf{a}_N = \kappa \left(\frac{ds}{dt}\right)^2 = 1 \cdot 4 = 4.$$

46. $\mathbf{r}'(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j} + t\sqrt{3} \mathbf{k}, \quad \frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{4t^2} = 2t, \quad \frac{d^2s}{dt^2} = 2$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2} \cos t \mathbf{i} + \frac{1}{2} \sin t \mathbf{j} + \frac{1}{2} \sqrt{3} \mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{1}{2} \sin t \mathbf{i} + \frac{1}{2} \cos t \mathbf{j}, \quad \|\mathbf{T}'(t)\| = \frac{1}{2}$$

Then, $\kappa = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{1/2}{2t} = \frac{1}{4t}$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = 2, \quad \mathbf{a}_N = \kappa \left(\frac{ds}{dt}\right)^2 = \frac{1}{4t} (4t^2) = t.$$

$$47. \quad \mathbf{r}'(t) = \sqrt{1+t} \mathbf{i} - \sqrt{1-t} \mathbf{j} + \sqrt{2} \mathbf{k}, \quad \frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{(1+t) + (1-t) + 2} = 2, \quad \frac{d^2s}{dt^2} = 0$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{1+t}}{2} \mathbf{i} - \frac{\sqrt{1-t}}{2} \mathbf{j} + \frac{\sqrt{2}}{2} \mathbf{k}$$

$$\mathbf{T}'(t) = \frac{1}{4\sqrt{1+t}} \mathbf{i} + \frac{1}{4\sqrt{1-t}} \mathbf{j}.$$

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{1}{16(1+t)} + \frac{1}{16(1-t)}} = \frac{1}{4} \sqrt{\frac{2}{1-t^2}}$$

$$\text{Then, } \kappa = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{1}{8} \sqrt{\frac{2}{1-t^2}}$$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = 0, \quad \mathbf{a}_N = \kappa \left(\frac{ds}{dt}\right)^2 = \frac{1}{2} \sqrt{\frac{2}{1-t^2}}.$$