CHAPTER 14

SECTION 14.1

3. dom (f) = the set of all points (x, y) except those on the line $y = -x$; range $(f) = (-\infty, 0) \cup (0, \infty)$

4. dom (f) = the set of all points (x, y) other than the origin; range $(f) = (0, \infty)$

7. dom (f) = the first and third quadrants, excluding the axes; range $(f) = (-\infty, \infty)$

9. dom (f) = the set of all points (x, y) with $x^2 < y$ —in other words, the set of all points of the plane above the parabola $y = x^2$; range $(f) = (0, \infty)$

10. dom (f) = the set of all points (x, y) with $-3 \le x \le 3$, $-1 \le y \le 1$ (a rectangle);

    range $(f) = [0, 3]$

15. dom (f) = the set of all points (x, y, z) with $|y| < |x|$; range $(f) = (-\infty, 0\,]$

17. dom (f) = the set of all points (x, y) with $x^2 + y^2 < 9$ —in other words, the set of all points of the plane inside the circle $x^2 + y^2 = 9$; range $(f) = [\,2/3, \infty)$

20. dom (f) = the set of all points (x, y, z) with $x^2 + y^2 + z^2 \le 4$— in other words, the set of all points inside and on the sphere $x^2 + y^2 + z^2 = 4$; range $(f) = [1, e^2]$

27. $\lim_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \dfrac{2(x+h)^2 - y - (2x^2 - y)}{h} = \lim_{h\to 0} \dfrac{4xh + 2h^2}{h} = 4x$

    $\lim_{h\to 0} \dfrac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \dfrac{2x^2 - (y+h) - (2x^2 - y)}{h} = -1$

29. $\lim_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \dfrac{3(x+h) - (x+h)y + 2y^2 - (3x - xy + 2y^2)}{h} = \lim_{h\to 0} \dfrac{3h - hy}{h} = 3 - y$

    $$\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \frac{3x - x(y+h) + 2(y+h)^2 - (3x - xy + 2y^2)}{h}$$
    $$= \lim_{h\to 0} \frac{-xh + 4yh + 2h^2}{h} = -x + 4y$$

30. $\lim_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \dfrac{x\sin y + h\sin y - x\sin y}{h} = \lim_{h\to 0} \sin y = \sin y.$

    $\lim_{h\to 0} \dfrac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \dfrac{x\sin(y+h) - x\sin y}{h} = x \lim_{h\to 0} \dfrac{\sin(y+h) - \sin y}{h} = x\cos y$

35. (a) $f(x, y) = x^2 y$ (b) $f(x, y) = \pi x^2 y$ (c) $f(x, y) = |\,2\,\mathbf{i} \times (x\,\mathbf{i} + y\,\mathbf{j})\,| = 2|\,y\,|$

36. (a) $f(x, y, z) = xy + 2xz + 2yz$

    (b) $f(x, y, z) = \cos^{-1} \dfrac{(\mathbf{i} + \mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\|\mathbf{i} + \mathbf{j}\| \|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|} = \cos^{-1} \dfrac{x + y}{\sqrt{2}\sqrt{x^2 + y^2 + z^2}}$

    (c) $f(x, y, z) = [\mathbf{i} \times (\mathbf{i} + \mathbf{j})] \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = z$

37. Surface area: $S = 2lw + 2lh + 2hw = 20 \implies w = \dfrac{20 - 2lh}{2l + 2h} = \dfrac{10 - lh}{l + h}$

Volume: $V = lwh = \dfrac{lh(10 - lh)}{l + h}$

38. $wlh = 12 \implies h = \dfrac{12}{wl}; \quad C = 4wl + 2(2wh + 2lh) = 4wl + \dfrac{48}{l} + \dfrac{48}{w}$

SECTION 14.2

45. $x + 2y + 3\left(\dfrac{x + y - 6}{2}\right) = 6$ or $5x + 7y = 30$, a line

48. $z^2 + (z - 2)^2 = 2 \implies 2(z - 1)^2 = 0 \implies z = 1 \implies x^2 + y^2 = 1$, a circle.

49. $x^2 + y^2 + \left(x^2 + 3y^2\right) = 4$ or $x^2 + 2y^2 = 2$, an ellipse

51. $x^2 + y^2 = (2 - y)^2$ or $x^2 = -4(y - 1)$, a parabola

SECTION 14.3

2. lines of slope 2 : $y = 2x - c$

3. parabolas: $y = x^2 - c$

7. the cubics $y = x^3 - c$

8. the coordinate axes and the hyperbolas $xy = \ln c$

13. the circles $x^2 + y^2 = e^c$, c real

14. the parabolas $y = e^c x^2$ with the origin omitted throughout

19. $x + 2y + 3z = 0$, plane through the origin

20. circular cylinder $x^2 + y^2 = 4$ (Figure 14.2.8)

21. $z = \sqrt{x^2 + y^2}$, the upper nappe of the circular cone $z^2 = x^2 + y^2$ (Figure 14.2.4)

22. ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{6} + \dfrac{z^2}{9} = 1$ (Figure 14.2.1)

27. The level curves of f are: $1 - 4x^2 - y^2 = c$. Substituting $P(0, 1)$ into this equation, we have

$$1 - 4(0)^2 - (1)^2 = c \implies c = 0$$

The level curve that contains P is: $1 - 4x^2 - y^2 = 0$, or $4x^2 + y^2 = 1$.

28. $(x^2 + y^2)e^{xy} = 1$

31. The level surfaces of f are: $x^2 + 2y^2 - 2xyz = c$. Substituting $P(-1, 2, 1)$ into this equation, we have

$$(-1)^2 + 2(2)^2 - 2(-1)(2)(1) = c \implies c = 13$$

The level surface that contains P is: $x^2 + 2y^2 - 2xyz = 13$.

37. $\dfrac{GmM}{x^2+y^2+z^2} = c \implies x^2+y^2+z^2 = \dfrac{GmM}{c}$; the surfaces of constant gravitational force are concentric spheres.

42. D.

43. $f(x,y) = \cos\sqrt{x^2+y^2}$; A

45. $f(x,y) = xye^{-(x^2+y^2)/2}$; E

SECTION 14.4

2. $\dfrac{\partial g}{\partial x} = 2xe^{-y}, \quad \dfrac{\partial g}{\partial y} = -x^2e^{-y}$

5. $\dfrac{\partial f}{\partial x} = e^{x-y} + e^{y-x}, \quad \dfrac{\partial f}{\partial y} = -e^{x-y} - e^{y-x}$

6. $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2-3y}}, \quad \dfrac{\partial z}{\partial y} = \dfrac{-3}{2\sqrt{x^2-3y}}$

8. $\dfrac{\partial u}{\partial x} = \dfrac{-e^z}{x^2y^2}, \quad \dfrac{\partial u}{\partial y} = \dfrac{-2e^z}{xy^3}, \quad \dfrac{\partial u}{\partial z} = \dfrac{e^z}{xy^2}$

9. $\dfrac{\partial u}{\partial x} = y+z, \quad \dfrac{\partial u}{\partial y} = x+z, \quad \dfrac{\partial u}{\partial z} = x+y$

11. $\dfrac{\partial f}{\partial x} = z\cos(x-y), \quad \dfrac{\partial f}{\partial y} = -z\cos(x-y), \quad \dfrac{\partial f}{\partial z} = \sin(x-y)$

12. $\dfrac{\partial g}{\partial u} = \dfrac{2u}{u^2+vw-w^2}, \quad \dfrac{\partial g}{\partial v} = \dfrac{w}{u^2+vw-w^2}, \quad \dfrac{\partial g}{\partial w} = \dfrac{v-2w}{u^2+vw-w^2}$

15. $\dfrac{\partial f}{\partial x} = 2xy\sec xy + x^2y(\sec xy)(\tan xy)y = 2xy\sec xy + x^2y^2\sec xy\,\tan xy$

$\dfrac{\partial f}{\partial y} = x^2\sec xy + x^2y(\sec xy)(\tan xy)x = x^2\sec xy + x^3y\sec xy\,\tan xy$

19. $\dfrac{\partial f}{\partial x} = \dfrac{(y\cos x)\sin y - (x\sin y)(-y\sin x)}{(y\cos x)^2} = \dfrac{\sin y(\cos x + x\sin x)}{y\cos^2 x}$

$\dfrac{\partial f}{\partial y} = \dfrac{(y\cos x)(x\cos y) - (x\sin y)\cos x}{(y\cos x)^2} = \dfrac{x(y\cos y - \sin y)}{y^2\cos x}$

20. $\dfrac{\partial f}{\partial x} = e^{xy}(y\sin xz + z\cos xz), \quad \dfrac{\partial f}{\partial y} = xe^{xy}\sin xz, \quad \dfrac{\partial f}{\partial z} = xe^{xy}\cos xz$

26. $\dfrac{\partial u}{\partial x} = \dfrac{1}{x} - yze^{xz}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{1}{y} - e^{xz}, \quad \dfrac{\partial u}{\partial z} = -xye^{xz}$

43. Let $z = f(x,y) = x^2 + y^2$. Then $f(2,1) = 5$, $f_y(x,y) = 2y$ and $f_y(2,1) = 2$;

equations for the tangent line are: $x = 2, \quad z - 5 = 2(y-1)$

45. Let $z = f(x,y) = \dfrac{x^2}{y^2-3}$. Then $f(3,2) = 9$, $f_x(x,y) = \dfrac{2x}{y^2-3}$ and $f_x(3,2) = 6$;

equations for the tangent line are: $y = 2, \quad z - 9 = 6(x-3)$

46. $f(x,y) = (4-x^2-y^2)^{1/2}, \quad f_x(x,y) = -x(4-x^2-y^2)^{-1/2}, \quad f_y(x,y) = -y(4-x^2-y^2)^{-1/2}$

(a) $f_y(1,1) = -\dfrac{\sqrt{2}}{2} \implies x = 1,\ z - \sqrt{2} = -\dfrac{\sqrt{2}}{2}(y-1)$

(b) $f_x(1,1) = -\dfrac{\sqrt{2}}{2} \implies y = 1,\ z - \sqrt{2} = -\dfrac{\sqrt{2}}{2}(x-1)$

(c) l_1 and l_2 have direction vectors $\mathbf{j}-\frac{\sqrt{2}}{2}\mathbf{k}$, $\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{k}$ respectively. The normal to the plane is

$$(\mathbf{j}-\frac{\sqrt{2}}{2}\mathbf{k})\times(\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{k}) = -\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j}-\mathbf{k}, \text{ so the tangent plane is}$$

$$-\frac{\sqrt{2}}{2}(x-1)-\frac{\sqrt{2}}{2}(y-1)-(z-\sqrt{2}) = 0, \quad \text{or} \quad (x-1)+(y-1)+\sqrt{2}(z-\sqrt{2}) = 0$$

53. (a) f depends only on y. (b) f depends only on x.

55. (a) $50\sqrt{3}$ in.^{2}

(b) $\dfrac{\partial A}{\partial b} = \dfrac{1}{2}c\sin\theta$; at time t_0, $\dfrac{\partial A}{\partial b} = 5\sqrt{3}$

(c) $\dfrac{\partial A}{\partial \theta} = \dfrac{1}{2}bc\cos\theta$; at time t_0, $\dfrac{\partial A}{\partial \theta} = 50$

(d) with $h = \dfrac{\pi}{180}$, $A(b,c,\theta+h)-A(b,c,\theta)\cong h\dfrac{\partial A}{\partial\theta} = \dfrac{\pi}{180}(50) = \dfrac{5\pi}{18}$ in.^{2}

(e) $0 = \dfrac{1}{2}\sin\theta\left(b\dfrac{\partial c}{\partial b}+c\right)$; at time t_0, $\dfrac{\partial c}{\partial b} = \dfrac{-c}{b} = -2$

SECTION 14.5

1. interior $= \{(x,y): 2<x<4, \quad 1<y<3\}$ (the inside of the rectangle), boundary = the union of the four boundary line segments; set is closed.

2. same interior and same boundary as in Exercise 1; set is open

3. interior = the entire set (region between the two concentric circles), boundary = the two circles, one of radius 1, the other of radius 2; set is open.

4. interior$=\{(x,y): 1<x^2<4\} = \{(x,y): -2<x<-1\}\cup\{(x,y): 1<x<2\}$ (two vertical stripes without the boundary lines), boundary$=\{(x,y): x=-2,\ x=-1,\ x=1, \text{or } x=2\}$ (four vertical lines); set is closed.

6. interior = the entire set (region below the parabola $y=x^2$), boundary = the parabola $y=x^2$; the set is open.

7. interior = region below the parabola $y=x^2$, boundary = the parabola $y=x^2$; the set is closed.

9. interior $= \{(x,y,z): x^2+y^2<1,\ 0<z\le 4\}$ (the inside of the cylinder), boundary = the total surface of the cylinder (the curved part, the top, and the bottom); the set is closed.

Section 14.6

2. $\dfrac{\partial^2 f}{\partial x^2} = 6Ax + 2By, \quad \dfrac{\partial^2 f}{\partial y^2} = 2Cx, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = 2Bx + 2Cy$

3. $\dfrac{\partial^2 f}{\partial x^2} = Cy^2 e^{xy}, \quad \dfrac{\partial^2 f}{\partial y^2} = Cx^2 e^{xy}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = Ce^{xy}(xy + 1)$

4. $\dfrac{\partial^2 f}{\partial x^2} = 2\cos y - y^2 \sin x, \quad \dfrac{\partial^2 f}{\partial y^2} = 2\sin x - x^2 \cos y, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = 2(y\cos x - x\sin y)$

7. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{(x+y)^2} - \dfrac{1}{x^2}, \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{1}{(x+y)^2}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{1}{(x+y)^2}$

13. $\dfrac{\partial^2 f}{\partial x^2} = ye^x, \quad \dfrac{\partial^2 f}{\partial y^2} = xe^y, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = e^y + e^x$

14. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}, \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{-2xy}{(x^2+y^2)^2}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

15. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}, \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = -\dfrac{2xy}{(x^2+y^2)^2}$

19. $\dfrac{\partial^2 f}{\partial x^2} = 0, \quad \dfrac{\partial^2 f}{\partial y^2} = xz\,\sin y, \quad \dfrac{\partial^2 f}{\partial z^2} = -xy\,\sin z,$

$\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \sin z - z\,\cos y, \quad \dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x} = y\,\cos z - \sin y, \quad \dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial^2 u}{\partial z \partial y} = x\,\cos z - x\,\cos y$

20. $\dfrac{\partial^2 f}{\partial x^2} = ze^x, \quad \dfrac{\partial^2 f}{\partial y^2} = xe^y, \quad \dfrac{\partial^2 f}{\partial z^2} = ye^z, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = e^y,$

$\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\partial^2 f}{\partial x \partial z} = e^x, \quad \dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{\partial^2 f}{\partial y \partial z} = e^z$

23. (a) no, since $\dfrac{\partial^2 f}{\partial y \partial x} \neq \dfrac{\partial^2 f}{\partial x \partial y}$ (b) no, since $\dfrac{\partial^2 f}{\partial y \partial x} \neq \dfrac{\partial^2 f}{\partial x \partial y}$ for $x \neq y$

26. (a) as (x, y) tends to $(0, 0)$ along the x-axis, $f(x, y) = f(x, 0) = 1$ tends to 1;

as (x, y) tends to $(0, 0)$ along the line $y = x$, $f(x, y) = f(x, x) = 0$ tends to 0;

(b) as (x, y) tends to $(0, 0)$ along the x-axis, $f(x, y) = f(x, 0) = 0$ tends to 0;

as (x, y) tends to $(0, 0)$ along the line $y = x$, $f(x, y) = f(x, x) = \frac{1}{2}$ tends to $\frac{1}{2}$;

27. (a) $\displaystyle\lim_{x\to 0} \frac{(x)(0)}{x^2 + 0} = \lim_{x\to 0} 0 = 0$ (b) $\displaystyle\lim_{y\to 0} \frac{(0)(y)}{0 + y^2} = \lim_{y\to 0} 0 = 0$

(c) $\displaystyle\lim_{x\to 0} \frac{(x)(mx)}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$

(d) $\lim_{\theta\to 0^+} \dfrac{(\theta\cos\theta)(\theta\sin\theta)}{(\theta\cos\theta)^2 + (\theta\sin\theta)^2} = \lim_{\theta\to 0^+} \cos\theta\,\sin\theta = 0$

(e) By L'Hospital's rule $\lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} f'(x) = f'(0)$. Thus

$$\lim_{x\to 0} \frac{xf(x)}{x^2 + [\,f(x)\,]^2} = \lim_{x\to 0} \frac{f(x)/x}{1 + [\,f(x)/x\,]^2} = \frac{f'(0)}{1 + [\,f'(0)\,]^2}.$$

(f) $\lim_{\theta\to(\pi/3)^-} = \dfrac{(\cos\theta\,\sin 3\theta)(\sin\theta\,\sin 3\theta)}{(\cos\theta\,\sin 3\theta)^2 + (\sin\theta\,\sin 3\theta)^2} = \lim_{\theta\to(\pi/3)^-} \cos\theta\,\sin\theta = \dfrac{1}{4}\sqrt{3}$

(g) $\lim_{t\to\infty} \dfrac{(1/t)(\sin t)/t}{1/t^2 + \left(\sin^2 t\right)/t^2} = \lim_{t\to\infty} \dfrac{\sin t}{1 + \sin^2 t}$; does not exist

28. (a) $\lim_{x\to 0} \dfrac{x(0)^2}{(x^2 + 0^2)^{3/2}} = \lim_{x\to 0} 0 = 0$ (b) $\lim_{y\to 0} \dfrac{0\cdot y^2}{(0 + y^2)^{3/2}} = \lim_{y\to 0} 0 = 0$

(c) $\lim_{x\to 0} \dfrac{xm^2x^2}{(x^2 + m^2x^2)^{3/2}} = \lim_{x\to 0} \dfrac{m^2x^3}{|x|^3(1 + m^2)^{3/2}} = \lim_{x\to 0} \dfrac{m^2x}{|x|\,(1 + m^2)^{3/2}}$; does not exist

(d) $\lim_{\theta\to 0^+} \dfrac{(\theta\cos\theta)(\theta\sin\theta)^2}{[(\theta\cos\theta)^2 + (\theta\sin\theta)^2]^{3/2}} = \lim_{\theta\to 0^+} \cos\theta\sin^2\theta = 0$

(e) $\lim_{x\to 0} \dfrac{x[f(x)]^2}{(x^2 + [f(x)]^2)^{3/2}} = \lim_{x\to 0} \dfrac{[f(x)/x]^2}{(1 + [f(x)/x]^2)^{3/2}} = \lim_{x\to 0} \dfrac{x^3[f'(0)]^2}{|x|^3(1 + [f'(0)]^2)^{3/2}}$; does not exist

(f) $\lim_{\theta\to\frac{\pi}{3}^-} \dfrac{(\cos\theta\sin 3\theta)(\sin\theta\sin 3\theta)^2}{[(\cos\theta\sin 3\theta)^2 + (\sin\theta\sin 3\theta)^2]^{3/2}} = \lim_{\theta\to\frac{\pi}{3}^-} \cos\theta\sin^2\theta = \dfrac{3}{8}$

(g) $\lim_{t\to\infty} \dfrac{(1/t)\,(\sin t/t)^2}{\left[(1/t^2) + (\sin^2 t/t^2)\right]^{3/2}} = \lim_{t\to\infty} \dfrac{\sin^2 t}{\left(1 + \sin^2 t\right)^{3/2}}$; does not exist