CHAPTER 15

SECTION 15.1

1. $\nabla f = (6x - y)\,\mathbf{i} + (1 - x)\,\mathbf{j}$

3. $\nabla f = e^{xy}[\,(xy + 1)\,\mathbf{i} + x^2\mathbf{j}]$

5. $\nabla f = \left[2y^2\sin(x^2+1) + 4x^2y^2\cos(x^2+1)\right]\,\mathbf{i} + 4xy\sin(x^2+1)\,\mathbf{j}$

6. $\nabla f = \dfrac{2x}{x^2+y^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2}\,\mathbf{j}$

9. $\nabla f = (z^2 + 2xy)\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + (y^2 + 2zx)\,\mathbf{k}$

11. $\nabla f = e^{-z}(2xy\,\mathbf{i} + x^2\,\mathbf{j} - xy^2\,\mathbf{k})$

12.
$$\nabla f = \left[\frac{xyz}{x+y+z} + yz\,\ln(x+y+z)\right]\,\mathbf{i} + \left[\frac{xyz}{x+y+z} + xz\,\ln(x+y+z)\right]\,\mathbf{j} + \left[\frac{xyz}{x+y+z} + xy\,\ln(x+y+z)\right]\,\mathbf{k}$$

15. $\nabla f = \left[2y\cos(2xy) + \dfrac{2}{x}\right]\,\mathbf{i} + 2x\cos(2xy)\,\mathbf{j} + \dfrac{1}{z}\,\mathbf{k}$

17. $\nabla f = (4x - 3y)\,\mathbf{i} + (8y - 3x)\,\mathbf{j};$ at $(2,3)$, $\nabla f = -\mathbf{i} + 18\mathbf{j}$

19. $\nabla f = \dfrac{2x}{x^2+y^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2}\,\mathbf{j};$ at $(2,1)$, $\nabla f = \dfrac{4}{5}\,\mathbf{i} + \dfrac{2}{5}\,\mathbf{j}$

20. $\nabla f = \left(\tan^{-1}(y/x) - \dfrac{xy}{x^2+y^2}\right)\mathbf{i} + \left(\dfrac{x^2}{x^2+y^2}\right)\mathbf{j},\quad \nabla f(1,1) = \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right)\mathbf{i} + \dfrac{1}{2}\,\mathbf{j}$

23. $\nabla f = -e^{-x}\sin(z + 2y)\,\mathbf{i} + 2e^{-x}\cos(z + 2y)\,\mathbf{j} + e^{-x}\cos(z + 2y)\,\mathbf{k};$

at $(0, \pi/4, \pi/4)$, $\quad \nabla f = -\frac{1}{2}\sqrt{2}\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

26. $\nabla f = -\sin(xyz^2)(yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}),\quad \nabla f\left(\pi, \dfrac{1}{4}, -1\right) = -\dfrac{\sqrt{2}}{2}\left(\dfrac{1}{4}\,\mathbf{i} + \pi\,\mathbf{j} - \dfrac{\pi}{2}\,\mathbf{k}\right)$

33. $\nabla f = \mathbf{F}(x,y) = 2xy\,\mathbf{i} + \left(1 + x^2\right)\,\mathbf{j} \quad\Rightarrow\quad \dfrac{\partial f}{\partial x} = 2xy \quad\Rightarrow\quad f(x,y) = x^2y + g(y)$ for some function g.

Now, $\dfrac{\partial f}{\partial y} = x^2 + g'(y) = 1 + x^2 \quad\Rightarrow\quad g'(y) = 1 \quad\Rightarrow\quad g(y) = y + C,$ C a constant.

Thus, $f(x,y) = x^2y + y$ (take $C = 0$) is a function whose gradient is $\mathbf{F}$.

34. $\nabla f = (2xy + x)\mathbf{i} + (x^2 + y)\mathbf{j} \implies f_x = 2xy + x,\quad f_y = x^2 + y$

Take $\quad f(x,y) = x^2y + \dfrac{x^2}{2} + \dfrac{y^2}{2}$

39. (a) $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} = \mathbf{0} \implies x = y = 0;\quad \nabla f = \mathbf{0}$ at $(0,0)$.

f has an absolute minimum at $(0,0)$

40. (a) $\nabla f = \dfrac{-1}{\sqrt{4-x^2-y^2}}(x\,\mathbf{i} + y\,\mathbf{j}) = \mathbf{0}$ at $(0,0)$

(c) f has a maximum at $(0,0)$

SECTION 15.2

3. $\nabla f\,(e^y - ye^x)\,\mathbf{i} + (xe^y - e^x)\,\mathbf{j}, \quad \nabla f(1,0) = \mathbf{i} + (1-e)\mathbf{j}, \quad \mathbf{u} = \dfrac{1}{5}(3\mathbf{i} + 4\mathbf{j}),$

$f'_{\mathbf{u}}(1,0) = \nabla f(1,0) \cdot \mathbf{u} = \dfrac{1}{5}(7-4e)$

4. $\nabla f = \dfrac{1}{(x-y)^2}(-2y\mathbf{i} + 2x\mathbf{j}), \quad \nabla f(1,0) = 2\mathbf{j}, \quad \mathbf{u} = \dfrac{1}{2}(\mathbf{i} - \sqrt{3}\mathbf{j}),$

$f'_{\mathbf{u}}(1,0) = \nabla f(1,0) \cdot \mathbf{u} = -\sqrt{3}$

7. $\nabla f = \dfrac{2x}{x^2+y^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2}\,\mathbf{j}, \quad \nabla f(0,1) = 2\,\mathbf{j}, \quad \mathbf{u} = \dfrac{1}{\sqrt{65}}(8\,\mathbf{i} + \mathbf{j}),$

$f'_{\mathbf{u}}(0,1) = \nabla f \cdot \mathbf{u} = \dfrac{2}{\sqrt{65}}$

9. $\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (y+x)\mathbf{k}, \quad \nabla f(1,-1,1) = 2\mathbf{j}, \quad \mathbf{u} = \tfrac{1}{6}\sqrt{6}\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$

$f'_{\mathbf{u}}(1,-1,1) = \nabla f(1,-1,1) \cdot \mathbf{u} = \dfrac{2}{3}\sqrt{6}$

10. $\nabla f = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}, \quad \nabla f(1,0,1) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{u} = \dfrac{1}{\sqrt{10}}(3\mathbf{i} - \mathbf{k})$

$f'_{\mathbf{u}}(1,0,1) = \nabla f(1,0,1) \cdot \mathbf{u} = \dfrac{\sqrt{10}}{10}$

15. $\nabla f = \dfrac{x}{x^2+y^2}\,\mathbf{i} + \dfrac{y}{x^2+y^2}\,\mathbf{j}, \quad \mathbf{u} = \dfrac{1}{\sqrt{x^2+y^2}}\,(-x\mathbf{i} - y\mathbf{j}), \quad f'_{\mathbf{u}}(x,y) = \nabla f \cdot \mathbf{u} = -\dfrac{1}{\sqrt{x^2+y^2}}$

18. $\nabla f = \dfrac{z}{x}\mathbf{i} - \dfrac{z}{y}\mathbf{j} + \ln\left(\dfrac{x}{y}\right)\mathbf{k}, \quad \nabla f(1,1,2) = 2\mathbf{i} - 2\mathbf{j}$

$\mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k}); \quad f'_{\mathbf{u}}(1,1,2) = \nabla f(1,1,2) \cdot \mathbf{u} = \mathbf{0}$

19. $\nabla f = e^{y^2-z^2}(\mathbf{i} + 2xy\mathbf{j} - 2xz\mathbf{k}), \quad \nabla f(1,2,-2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}, \quad \mathbf{r}'(t) = \mathbf{i} - 2\,\sin(t-1)\,\mathbf{j} - 2e^{t-1}\mathbf{k},$

at $(1,2,-2)$ $\quad t = 1, \quad \mathbf{r}'(1) = \mathbf{i} - 2\mathbf{k}, \quad \mathbf{u} = \tfrac{1}{5}\sqrt{5}\,(\mathbf{i} - 2\mathbf{k}), \quad f'_{\mathbf{u}}(1,2,-2) = \nabla f(1,2,-2) \cdot \mathbf{u} = -\dfrac{7}{5}\sqrt{5}$

20. $\nabla f = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}, \quad \nabla f(1,-3,2) = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

Direction: $\mathbf{r}'(-1) = -2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $\quad \mathbf{u} = \dfrac{1}{\sqrt{22}}(-2\mathbf{i}+3\mathbf{j}-3\mathbf{k})$, $\quad f'_{\mathbf{u}}(1,-3,2) = \nabla f(1,-3,2)\cdot\mathbf{u} = \dfrac{1}{2}\sqrt{22}$

21. $\nabla f = (2x+2yz)\,\mathbf{i} + \left(2xz - z^2\right)\,\mathbf{j} + (2xy-2yz)\,\mathbf{k}, \quad \nabla f(1,1,2) = 6\,\mathbf{i} - 2\,\mathbf{k}$

The vector $\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ is a direction vector for the given line; $\mathbf{u} = \dfrac{1}{\sqrt{14}}(2\,\mathbf{i}+\mathbf{j}-3\,\mathbf{k})$

is a corresponding unit vector; $\quad f'_{\mathbf{u}}(1,1,2) = \nabla f(1,1,2)\cdot\mathbf{u} = \dfrac{18}{\sqrt{14}}$

23. $\nabla f = 2y^2e^{2x}\,\mathbf{i} + 2ye^{2x}\,\mathbf{j}, \quad \nabla f(0,1) = 2\,\mathbf{i} + 2\,\mathbf{j}, \quad \|\nabla f\| = 2\sqrt{2}, \quad \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$

f increases most rapidly in the direction $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$; the rate of change is $2\sqrt{2}$.

f decreases most rapidly in the direction $\mathbf{v} = -\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$; the rate of change is $-2\sqrt{2}$.

26. $\nabla f = (2xze^y + z^2)\mathbf{i} + x^2ze^y\mathbf{j} + (x^2e^y + 2xz)\mathbf{k} \quad \nabla f(1,\ln 2,2) = 12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

Fastest increase in direction $\mathbf{u} = \dfrac{1}{7}(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})$, $\quad$ rate of change $\|\nabla f(1,\ln 2,2)\| = 14$

Fastest decrease in direction $\mathbf{v} = -\dfrac{1}{7}(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})$, $\quad$ rate of change -14

33. (a) The projection of the path onto the xy-plane is the curve

$$C\colon\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

which begins at $(1,1)$ and at each point has its tangent vector in the direction of $-\nabla f$.

Since

$$\nabla f = 2x\mathbf{i} + 6y\mathbf{j},$$

we have the initial-value problems

$$x'(t) = -2x(t), \quad x(0) = 1 \qquad \text{and} \qquad y'(t) = -6y(t), \quad y(0) = 1.$$

From Theorem 7.6.1 we find that

$$x(t) = e^{-2t} \qquad \text{and} \qquad y(t) = e^{-6t}.$$

Eliminating the parameter t, we find that C is the curve $y = x^3$ from $(1,1)$ to $(0,0)$.

(b) Here

$$x'(t) = -2x(t), \quad x(0) = 1 \qquad \text{and} \qquad y'(t) = -6y(t), \quad y(0) = -2$$

so that

$$x(t) = e^{-2t} \qquad \text{and} \qquad y(t) = -2e^{-6t}.$$

Eliminating the parameter t, we find that the projection of the path onto the xy-plane is the curve $y = -2x^3$ from $(1,-2)$ to $(0,0)$.

34. $z: f(x,y) = \dfrac{1}{2}x^2 - y^2$; $\nabla f = x\mathbf{i} - 2y\mathbf{j}$, so the projection $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ of the path onto the xy-plane must satisfy $x'(t) = x(t),\ y(t) = -2y(t)$

(a) With initial point $(-1, 1, -\frac{1}{2})$,, we get $x(t) = -e^t,\ y(t) = e^{-2t}$, or $y = \dfrac{1}{x^2}$ from $(-1, 1)$, in the direction of decreasing x.

(b) With initial point $(1, 0, \frac{1}{2})$, we get $x(t) = e^t,\ y(t) = 0$, or the x-axis from $(1, 0)$, in the direction of increasing x.

39. (a)

$$\lim_{h\to 0} \frac{f\big(2+h,\, (2+h)^2\big) - f(2,4)}{h} = \lim_{h\to 0} \frac{3(2+h)^2 + (2+h)^2 - 16}{h}$$

$$= \lim_{h\to 0} 4\left[\frac{4h + h^2}{h}\right] = \lim_{h\to 0} 4(4+h) = 16$$

(b)

$$\lim_{h\to 0} \frac{f\left(\dfrac{h+8}{4},\, 4+h\right) - f(2,4)}{h} = \lim_{h\to 0} \frac{3\left(\dfrac{h+8}{4}\right)^2 + (4+h) - 16}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{16}h^2 + 3h + 12 + 4 + h - 16}{h}$$

$$= \lim_{h\to 0} \left(\tfrac{3}{16}h + 4\right) = 4$$

(c) $\mathbf{u} = \frac{1}{17}\sqrt{17}\,(\mathbf{i} + 4\mathbf{j})$, $\nabla f(2,4) = 12\mathbf{i} + \mathbf{j}$; $f'_{\mathbf{u}}(2,4) = \nabla f(2,4) \cdot \mathbf{u} = \frac{16}{17}\sqrt{17}$

(d) The limits computed in (a) and (b) are not directional derivatives. In (a) and (b) we have, in essence, computed $\nabla f(2,4) \cdot \mathbf{r}_0$ taking $\mathbf{r}_0 = \mathbf{i} + 4\mathbf{j}$ in (a) and $\mathbf{r}_0 = \frac{1}{4}\mathbf{i} + \mathbf{j}$ in (b). In neither case is $\mathbf{r}_0$ a unit vector.

40. $\nabla f = \dfrac{-GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \dfrac{-GMm}{\|\mathbf{r}\|^3}\mathbf{r}$

SECTION 15.3

1. $f(\mathbf{b}) = f(1,3) = -2$; $f(\mathbf{a}) = f(0,1) = 0$; $f(\mathbf{b}) - f(\mathbf{a}) = -2$

$\nabla f = \big(3x^2 - y\big)\,\mathbf{i} - x\,\mathbf{j}$; $\mathbf{b} - \mathbf{a} = \mathbf{i} + 2\,\mathbf{j}$ and $\nabla f \cdot (\mathbf{b} - \mathbf{a}) = 3x^2 - y - 2x$

The line segment joining **a** and **b** is parametrized by

$$x = t, \quad y = 1 + 2t, \quad 0 \le t \le 1$$

Thus, we need to solve the equation

$$3t^2 - (1+2t) - 2t = -2, \quad \text{which is the same as} \quad 3t^2 - 4t + 1 = 0, \quad 0 \le t \le 1$$

The solutions are: $t = \frac{1}{3}, t = 1$. Thus, $\mathbf{c} = (\frac{1}{3}, \frac{5}{3})$ satisfies the equation.

Note that the endpoint **b** also satisfies the equation.

2. $\nabla f = 4z\mathbf{i} - 2y\mathbf{j} + (4x+2z)\mathbf{k}, \quad f(\mathbf{a}) = f(0,1,1) = 0, \quad f(\mathbf{b}) = f(1,3,2) = 3$

$\mathbf{b} - \mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k},$ so we want (x,y,z) such that

$\nabla f \cdot (\mathbf{b} - \mathbf{a}) = 4z - 4y + 4x + 2z = 6z - 4y + 4x = f(\mathbf{b}) - f(\mathbf{a}) = 3$

Parameterizing the line segment from **a** to **b** by $x(t) = t, \quad y(t) = 1 + 2t, \quad z(t) = 1 + t,$

we get $t = \frac{1}{2}$, or $\mathbf{c} = (\frac{1}{2}, 2, \frac{3}{2})$

7. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j};$

$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left(2\mathbf{i} + e^{2t}\mathbf{j}\right) \cdot (e^t\mathbf{i} - e^{-t}\mathbf{j}) = e^t$

10. $\nabla f = \dfrac{1}{2x^2 + y^3}(4x\mathbf{i} + 3y^2\mathbf{j})$

$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \dfrac{1}{2e^{4t} + t}(4e^{2t}\mathbf{i} + 3t^{2/3}\mathbf{j}) \cdot (2e^{2t}\mathbf{i} + \dfrac{1}{3}t^{-2/3}\mathbf{j}) = \dfrac{8e^{4t} + 1}{2e^{4t} + 1}$

11. $\nabla f = (e^y - ye^{-x})\,\mathbf{i} + (xe^y + e^{-x})\,\mathbf{j}; \quad \nabla f(\mathbf{r}(t)) = (t^t - \ln t)\,\mathbf{i} + \left(t^t \ln t + \dfrac{1}{t}\right)\mathbf{j}$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left((t^t - \ln t)\,\mathbf{i} + \left(t^t \ln t + \frac{1}{t}\right)\mathbf{j}\right) \cdot \left(\frac{1}{t}\,\mathbf{i} + [1 + \ln t]\,\mathbf{j}\right) = t^t\left(\frac{1}{t} + \ln t + [\ln t]^2\right) + \frac{1}{t}$$

13. $\nabla f = y\mathbf{i} + (x - z)\mathbf{j} - y\mathbf{k};$

$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left(t^2\mathbf{i} + \left(t - t^3\right)\mathbf{j} - t^2\mathbf{k}\right) \cdot \left(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}\right) = 3t^2 - 5t^4$

15. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k};$

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (2a\cos\omega t\,\mathbf{i} + 2b\sin\omega t\,\mathbf{j} + \mathbf{k}) \cdot (-a\omega\sin\omega t\,\mathbf{i} + b\omega\cos\omega t\,\mathbf{j} + b\omega\mathbf{k}) \\ &= 2\omega\left(b^2 - a^2\right)\sin\omega t\,\cos\omega t + b\omega\end{aligned}$$

17. $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = (2x - 3y)(-\sin t) + (4y - 3x)(\cos t)$

$= 2\cos t\sin t + 3\sin^2 t - 3\cos^2 t = \sin 2t - 3\cos 2t$

19. $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$

$= (e^x\sin y + e^y\cos x)\left(\frac{1}{2}\right) + (e^x\cos y + e^y\sin x)\,(2)$

$= e^{t/2}\left(\frac{1}{2}\sin 2t + 2\cos 2t\right) + e^{2t}\left(\frac{1}{2}\cos\frac{1}{2}t + 2\sin\frac{1}{2}t\right)$

20. $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = (4x - y)(-2\sin 2t) + (2y - x)\cos t$

$= 2\sin 2t(\sin t - 4\cos 2t) + \cos t(2\sin t - \cos 2t)$

23. $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$= (y + z)(2t) + (x + z)(1 - 2t) + (y + x)(2t - 2)$

$$= (1-t)(2t) + (2t^2 - 2t + 1)(1-2t) + t(2t-2)$$

$$= 1 - 4t + 6t^2 - 4t^3$$

29. $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x-y)(\cos t) + (-x)(t\cos s)$$

$$= 2s\cos^2 t - t\sin s\cos t - st\cos s\cos t$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = (2x-y)(-s\sin t) + (-x)(\sin s)$$

$$= -2s^2\cos t\sin t + st\sin s\sin t - s\cos t\sin s$$

32. $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= z^2 y\sec xy\tan xy(2t) + z^2 x\sec xy\tan xy + 2z\sec xy(2st)$$

$$= \sec[2st(s-t^2)]\left(2s^4t^3(s-t^2)\tan[2st(s-t^2)] + 2s^3t^2\tan[2st(s-t^2)] + 4s^3t^2\right)$$

$$\frac{\partial u}{\partial t} = z^2 y\sec xy\tan xy(2s) + z^2 x\sec xy\tan xy(-2t) + 2z\sec xy(s^2)$$

$$= \sec[2st(s-t^2)]\left(2s^5t^2(s-t^2)\tan[2st(s-t^2)] - 4s^5t^4\tan[2st(s-t^2)] + 2s^4t\right)$$

33. $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x-y)(\cos t) + (-x)(-\cos(t-s)) + 2z(t\cos s)$$

$$= 2s\cos^2 t - \sin(t-s)\cos t + s\cos t\cos(t-s) + 2t^2\sin s\cos s$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= (2x-y)(-s\sin t) + (-x)(\cos(t-s)) + 2z(\sin s)$$

$$= -2s^2\cos t\sin t + s\sin(t-s)\sin t - s\cos t\cos(t-s) + 2t\sin^2 s$$

35. $$\frac{d}{dt}\left[f(\mathbf{r}(t))\right] = \left[\nabla f(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\right]\|\mathbf{r}'(t)\|$$

$$= f'_{\mathbf{u}(t)}(\mathbf{r}(t))\,\|\mathbf{r}'(t)\| \quad \text{where} \quad \mathbf{u}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

SECTION 15.4

1. Set $f(x,y) = x^2 + xy + y^2$. Then,

$$\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j}, \quad \nabla f(-1,-1) = -3\mathbf{i} - 3\mathbf{j}.$$

normal vector $\mathbf{i}+\mathbf{j}$; tangent vector $\mathbf{i}-\mathbf{j}$

tangent line $x+y+2=0$; normal line $x-y=0$

2. Set $f(x,y) = (y-x)^2 - 2x$, $\nabla f = -2(y-x+1)\mathbf{i} + 2(y-x)\mathbf{j}$, $\nabla f(2,4) = -6\mathbf{i} + 4\mathbf{j}$

normal vector $-3\mathbf{i} + 2\mathbf{j}$; tangent vector $2\mathbf{i} + 3\mathbf{j}$

tangent line $3x - 2y + 2 = 0$; normal line $2x + 3y - 16 = 0$

5. Set $f(x,y) = xy^2 - 2x^2 + y + 5x$. Then,

$$\nabla f = (y^2 - 4x + 5)\,\mathbf{i} + (2xy + 1)\,\mathbf{j}, \quad \nabla f(4,2) = -7\mathbf{i} + 17\mathbf{j}.$$

normal vector $7\mathbf{i} - 17\mathbf{j}$; tangent vector $17\mathbf{i} + 7\mathbf{j}$

tangent line $7x - 17y + 6 = 0$; normal line $17x + 7y - 82 = 0$

7. Set $f(x,y) = 2x^3 - x^2y^2 - 3x + y$. Then,

$$\nabla f = (6x^2 - 2xy^2 - 3)\,\mathbf{i} + (-2x^2y + 1)\,\mathbf{j}, \quad \nabla f(1,-2) = -5\mathbf{i} + 5\mathbf{j}.$$

normal vector $\mathbf{i} - \mathbf{j}$; tangent vector $\mathbf{i} + \mathbf{j}$

tangent line $x - y - 3 = 0$; normal line $x + y + 1 = 0$

11. Set $f(x,y,z) = x^3 + y^3 - 3xyz$. Then,

$$\nabla f = (3x^2 - 3yz)\,\mathbf{i} + (3y^2 - 3xz)\,\mathbf{j} - 3xy\mathbf{k}, \quad \nabla f\left(1, 2, \tfrac{3}{2}\right) = -6\mathbf{i} + \tfrac{15}{2}\mathbf{j} - 6\mathbf{k};$$

tangent plane at $\left(1, 2, \frac{3}{2}\right)$: $-6(x-1) + \frac{15}{2}(y-2) - 6\left(z - \frac{3}{2}\right) = 0$, which reduces to $4x - 5y + 4z = 0$.

Normal: $x = 1 + 4t$, $y = 2 - 5t$, $z = \frac{3}{2} + 4t$

12. Set $f(x,y,z) = xy^2 + 2z^2$. $\nabla f = y^2\mathbf{i} + 2xy\mathbf{j} + 4z\mathbf{k}$, $\nabla f(1,2,2) = 4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$

Tangent plane: $x + y + 2z - 7 = 0$

Normal: $x = 1 + t$, $y = 2 + t$, $z = 2 + 2t$

14. Set $f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. $\nabla f = \dfrac{1}{2\sqrt{x}}\mathbf{i} + \dfrac{1}{2\sqrt{y}}\mathbf{j} + \dfrac{1}{2\sqrt{z}}\mathbf{k}$, $\nabla f(1,4,1) = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{4}\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

Tangent plane: $2x + y + 2z - 8 = 0$

Normal: $x = 1 + 2t$, $y = 4 + t$, $z = 1 + 2t$

16. Set $f(x,y,z) = x^2 + xy + y^2 - 6x + 2 - z$. $\nabla f = (2x + y - 6)\mathbf{i} + (x + 2y)\mathbf{j} - \mathbf{k}$, $\nabla f(4,-2,-10) = -\mathbf{k}$

Tangent plane: $z = -10$

Normal: $x = 4$, $y = -2$, $z = t$

20. $z = g(x,y) = 4x + 2y - x^2 + xy - y^2$. $\nabla g = (4 - 2x + y)\mathbf{i} + (2 + x - 2y)\mathbf{j}$

$\nabla g = \mathbf{0} \implies 4 - 2x + y = 0,\ 2 + x - 2y = 0 \implies x = \dfrac{10}{3},\ y = \dfrac{8}{3}$

The tangent plane is horizontal at $\left(\frac{10}{3}, \frac{8}{3}, \frac{28}{3}\right)$

23. Set $z = g(x,y) = 2x^2 + 2xy - y^2 - 5x + 3y - 2$. Then,

$$\nabla g = (4x + 2y - 5)\,\mathbf{i} + (2x - 2y + 3)\,\mathbf{j}.$$

$$\nabla g = \mathbf{0} \quad \Longrightarrow \quad 4x + 2y - 5 = 0 = 2x - 2y + 3 \quad \Longrightarrow \quad x = \tfrac{1}{3}, \quad y = \tfrac{11}{6}.$$

The tangent plane is horizontal at $\left(\frac{1}{3}, \frac{11}{6}, -\frac{1}{12}\right)$.

25. $$\frac{x - x_0}{(\partial f/\partial x)(x_0, y_0, z_0)} = \frac{y - y_0}{(\partial f/\partial y)(x_0, y_0, z_0)} = \frac{z - z_0}{(\partial f/\partial z)(x_0, y_0, z_0)}$$

33. Set $f(x, y, z) = x^2y^2 + 2x + z^3$. Then,

$$\nabla f = (2xy^2 + 2)\,\mathbf{i} + 2x^2y\mathbf{j} + 3z^2\mathbf{k}, \quad \nabla f(2, 1, 2) = 6\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}.$$

The plane tangent to $f(x, y, z) = 16$ at $(2, 1, 2)$ has equation

$$6(x - 2) + 8(y - 1) + 12(z - 2) = 0, \quad \text{or} \quad 3x + 4y + 6z = 22.$$

Next, set $g(x, y, z) = 3x^2 + y^2 - 2z$. Then,

$$\nabla g = 6x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}, \quad \nabla g(2, 1, 2) = 12\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

The plane tangent to $g(x, y, z) = 9$ at $(2, 1, 2)$ is

$$12(x - 2) + 2(y - 1) - 2(z - 2) = 0, \quad \text{or} \quad 6x + y - z = 11.$$

34. Sphere: $f(x, y, z) = z^2 + y^2 + z^2 - 8x - 8y - 6z + 24$, $\nabla f = (2x - 8)\mathbf{i} + (2y - 8)\mathbf{j} + (2z - 6)\mathbf{k}$

$\nabla f(2, 1, 1) = -4\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$

Ellipsoid: $g(x, y, z) = x^2 + 3y^2 + 2z^2$, $\nabla g = 2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}$

$\nabla g(2, 1, 1) = 4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

Since their normal vectors are parallel, the surfaces are tangent.

35. The gradient to the sphere at $(1, 1, 2)$ is

$$2x\mathbf{i} + (2y - 4)\,\mathbf{j} + (2z - 2)\mathbf{k} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

The gradient to the paraboloid at $(1, 1, 2)$ is

$$6x\mathbf{i} + 4y\mathbf{j} - 2\mathbf{k} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

Since

$$(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \cdot (6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 0,$$

the surfaces intersect at right angles.

SECTION 15.5

2. $\nabla f = (2 - 2x)\,\mathbf{i} + (2 + 2y)\,\mathbf{j} = \mathbf{0}$ only at $(1, -1)$.

The difference

$f(1 + h, -1 + k) - f(1, -1)$

$$= [2(1 + h) + 2(-1 + k) - (1 + h)^2 + (-1 + k)^2 + 5] - 5 = -h^2 + k^2$$

does not keep a constant sign for small h and k; $(1, -1)$ is a saddle point.

3. $\nabla f = (2x + y + 3)\,\mathbf{i} + (x + 2y)\,\mathbf{j} = \mathbf{0}$ only at $(-2, 1)$.

The difference

$f(-2 + h,\, 1 + k) - f(-2, 1)$

$$= [(-2 + h)^2 + (-2 + h)(1 + k) + (1 + k)^2 + 3(-2 + h) + 1] - (-2) = h^2 + hk + k^2$$

is positive for all small h and k. To see this, note that

$$h^2 + hk + k^2 \geq h^2 + k^2 - |h|\,|k| > 0;$$

there is a local minimum of -2 at $(-2, 1)$.

5. $\mathbf{\nabla} f = (2x + y - 6)\,\mathbf{i} + (x + 2y)\,\mathbf{j} = \mathbf{0}$ only at $(4, -2)$.

$f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$

At $(4, -2)$, $D = 3 > 0$ and $A = 2 > 0$ so we have a local min; the value is -10.

7. $\mathbf{\nabla} f = (3x^2 - 6y)\mathbf{i} + \left(3y^2 - 6x\right)\mathbf{j} = \mathbf{0}$ at $(2, 2)$ and $(0, 0)$.

$f_{xx} = 6x, \quad f_{xy} = -6, \quad f_{yy} = 6y, \quad D = 36xy - 36.$

At $(2, 2)$, $D = 108 > 0$ and $A = 12 > 0$ so we have a local min; the value is -8.

At $(0, 0)$, $D = -36 < 0$ so we have a saddle point.

9. $\mathbf{\nabla} f = (3x^2 - 6y + 6)\mathbf{i} + (2y - 6x + 3)\,\mathbf{j} = \mathbf{0}$ at $\left(5, \frac{27}{2}\right)$ and $\left(1, \frac{3}{2}\right)$.

$f_{xx} = 6x, \quad f_{xy} = -6, \quad f_{yy} = 2, \quad D = 12x - 36.$

At $\left(5, \frac{27}{2}\right)$, $D = 24 > 0$ and $A = 30 > 0$ so we have a local min; the value is $-\frac{117}{4}$.

At $\left(1, \frac{3}{2}\right)$, $D = -24 < 0$ so we have a saddle point.

10. $\mathbf{\nabla} f = (2x - 2y - 3)\mathbf{i} + (-2x + 4y + 5)\mathbf{j} = \mathbf{0} \quad$ at $\quad (\frac{1}{2}, -1)$
$\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y \partial x} = -2, \quad \dfrac{\partial^2 f}{\partial y^2} = 4; \quad D = 2 \cdot 4 - (-2)^2 > 0, \; A = 2 \implies$ local minimum;
the value is $-\frac{13}{4}$.

13. $\mathbf{\nabla} f = (2xy + 1 + y^2)\mathbf{i} + \left(x^2 + 2xy + 1\right)\mathbf{j} = \mathbf{0} \quad$ at $\quad (1, -1)$ and $(-1, 1)$.

$f_{xx} = 2y, \quad f_{xy} = 2x + 2y, \quad f_{yy} = 2x, \quad D = 4xy - 4(x + y)^2.$

At both $(1, -1)$ and $(-1, 1)$ we have saddle points since $D = -4 < 0$.

14. $\mathbf{\nabla} f = \left(\dfrac{1}{y} + \dfrac{y}{x^2}\right)\mathbf{i} + \left(-\dfrac{x}{y^2} - \dfrac{1}{x}\right)\mathbf{j} = \dfrac{x^2 + y^2}{x^2 y}\,\mathbf{i} - \dfrac{x^2 + y^2}{x y^2}\,\mathbf{j}$ is never $\mathbf{0}$;
no stationary points, no local extreme values.

20. $\mathbf{\nabla} f = \left(\ln xy + 1 - \dfrac{3}{x}\right)\mathbf{i} + \dfrac{x - 3}{y}\mathbf{j} = \mathbf{0} \quad$ at $\; (3,\, 1/3)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x} + \frac{3}{x^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{y}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{3 - x}{y^2}$$

At $\; (3,\, 1/3)$, $\quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{2}{3}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = 3, \quad \dfrac{\partial^2 f}{\partial y^2} = 0 \quad$ and $\quad D = -9 < 0 \implies$ saddle point.

21. $\mathbf{\nabla} f = \left(4x^3 - 4x\right)\mathbf{i} + 2y\,\mathbf{j} = \mathbf{0} \quad$ at $\quad (0, 0), \;\; (1, 0), \;$ and $\; (-1, 0)$.

$f_{xx} = 12x^2 - 4, \quad f_{xy} = 0, \quad f_{yy} = 2, \quad D = 8 - 24x^2.$

point	A	B	C	D	result
$(0,0)$	-4	0	2	-8	saddle
$(1,0)$	8	0	2	16	loc. min.
$(-1,0)$	8	0	2	16	loc. min.

$f(\pm 1, 0) = -3.$

25. (a) $\nabla f = (2x+ky)\,\mathbf{i} + (2y+kx)\,\mathbf{j}$ and $\nabla f(0,0) = \mathbf{0}$ independent of the value of k.

(b) $f_{xx} = 2, \quad f_{xy} = k, \quad f_{yy} = 2, \quad D = 4 - k^2$. Thus, $D < 0$ for $|k| > 2$ and $(0,0)$ is a saddle point

(c) $D = 4 - k^2 > 0$ for $|k| < 2$. Since $A = f_{xx} = 2 > 0$, $(0,0)$ is a local minimum.

(d) The test is inconclusive when $D = 4 - k^2 = 0$ i.e., for $k = \pm 2$.

26. (a) $\nabla f = (2x+ky)\mathbf{i} + (kx+8y)\mathbf{j} = \mathbf{0}$ at $(0,0)$.

(b) $\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y \partial x} = k, \quad \dfrac{\partial^2 f}{\partial y^2} = 8;$ we want $16 - k^2 < 0$, or $|k| > 4$

(c) We want $16 - k^2 > 0$, or $|k| < 4$

(d) $k = \pm 4$

27. Let $P\,(x,y,z)$ be a point in the plane. We want to find the minimum of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. However, it is sufficient to minimize the square of the distance: $F(x,y,z) = x^2+y^2+z^2$. It is clear that F has a minimum value, but no maximum value. Since P lies in the plane, $2x - y + 2z = 16$ which implies $y = 2x + 2z - 16 = 2(x+z-8)$. Thus, we want to find the minimum value of

$$F(x,z) = x^2 + 4(x+z-8)^2 + z^2$$

Now,

$$\nabla F = [2x + 8(x+z-8)]\,wi + [8(x+z-8)\,\mathbf{k}$$

The gradient is $\mathbf{0}$ when

$$2x + 8(x+z-8) = 0 \quad \text{and} \quad 8(x+z-8) + 2z = 0$$

The only solution to this pair of equations is: $x = z = \dfrac{32}{9}$, from which it follows that $y = -\dfrac{16}{9}$.

The point in the plane that is closest to the origin is $P\,\left(\frac{32}{9}, -\frac{16}{9}, \frac{32}{9}\right)$.

The distance from the origin to the plane is: $F(P) = \frac{16}{3}$.

Check using (12.6.7): $d(P,0) = \dfrac{|2\cdot 0 - 0 + 2\cdot 0 - 16|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \dfrac{16}{3}.$

SECTION 15.6

1. $\nabla f = (4x-4)\,\mathbf{i} + (2y-2)\,\mathbf{j} = \mathbf{0}$ at $(1,1)$ in D;

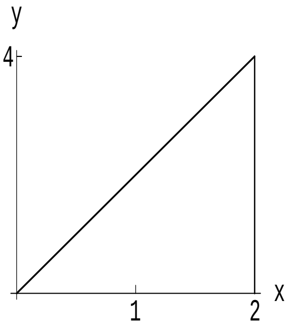
$f(1,1) = -1$

Next we consider the boundary of D. We parametrize each side of the triangle:

$C_1: \ \mathbf{r}_1(t) = t\,\mathbf{i}, \quad t \in [\,0,2\,],$

$C_2: \ \mathbf{r}_2(t) = 2\,\mathbf{i} + t\,\mathbf{j}, \quad t \in [\,0,4\,],$

$C_3: \ \mathbf{r}_3(t) = t\,\mathbf{i} + 2t\,\mathbf{j}, \quad t \in [\,0,2\,],$



Now,

$$f_1(t) = f(\mathbf{r}_1(t)) = 2(t-1)^2, \quad t \in [\,0,2\,]; \quad \text{critical number: } \ t = 1,$$

$$f_2(t) = f(\mathbf{r}_2(t)) = (t-1)^2 + 1, \quad t \in [\,0,4\,]; \quad \text{critical number: } \ t = 1,$$

$$f_3(t) = f(\mathbf{r}_3(t)) = 6t^2 - 8t + 2, \quad t \in [\,0,2\,]; \quad \text{critical number: } \ t = \tfrac{2}{3}.$$

Evaluating these functions at the endpoints of their domains and at the critical numbers, we find that:

$$f_1(0) = f_3(0) = f(0,0) = 2; \qquad f_1(1) = f(1,0) = 0; \qquad f_1(2) = f_2(0) = f(2,0) = 2;$$

$$f_2(1) = f(2,1) = 2; \qquad f_2(4) = f_3(2) = f(2,4) = 10; \qquad f_3(2/3) = f(2/3, 4/3) = -\tfrac{2}{3}.$$

f takes on its absolute maximum of 10 at $(2,4)$ and its absolute minimum of -1 at $(1,1)$.

5. $\nabla f = (2x+3y)\,\mathbf{i} + (2y+3x)\,\mathbf{j} = \mathbf{0}$ at $(0,0)$ in D; $\quad f(0,0) = 2$

Next we consider the boundary of D. We parametrize the circle by:

$C: \ \mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t \in [\,0, 2\pi\,]$

The values of f on the boundary are given by the function

$$F(t) = f(\mathbf{r}(t)) = 6 + 12\sin t\,\cos t, \quad t \in [\,0, 2\pi\,]$$

$$F'(t) = 12\cos^2 t - 12\sin^2 t: \quad F'(t) = 0 \quad \Longrightarrow \quad \cos t = \pm\,\sin t \quad \Longrightarrow \quad t = \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi$$

Evaluating F at the endpoints and critical numbers, we have:

$$F(0) = F(2\pi) = f(2,0) = 6; \quad F\left(\tfrac{1}{4}\pi\right) = F\left(\tfrac{5}{4}\pi\right) = f\left(\sqrt{2}, \sqrt{2}\right) = f(\left(-\sqrt{2}, -\sqrt{2}\right) = 12;$$

$$F\left(\tfrac{3}{4}\pi\right) = f\left(-\sqrt{2}, \sqrt{2}\right) = F\left(\tfrac{7}{4}\pi\right) = f(\left(\sqrt{2}, -\sqrt{2}\right) = 0$$

f takes on its absolute maximum of 12 at $\left(\sqrt{2}, \sqrt{2}\right)$ and at $\left(-\sqrt{2}, -\sqrt{2}\right)$; f takes on its absolute minimum of 0 at $\left(-\sqrt{2}, \sqrt{2}\right)$ and at $\left(\sqrt{2}, -\sqrt{2}\right)$.

8. $\nabla f = (y+1)\,\mathbf{i} + (x-1)\,\mathbf{j} = \mathbf{0}$ at $(1,-1)$ which is not in the interior of D.

Next we consider the boundary of D. We parametrize the boundary by:

$$C_1: \ \mathbf{r}_1(t) = t\,\mathbf{j} + t^2\,\mathbf{j}, \quad t \in [\,-2, 2\,],$$

$$C_2: \ \mathbf{r}_2(t) = t\,\mathbf{i} + 4\,\mathbf{j}, \quad t \in [\,-2, 2\,],$$

and evaluate f:

$$f_1(t) = f(\mathbf{r}_1(t)) = t^3 - t^2 + t + 3, \quad t \in [\,-2, 2\,]; \quad \text{no critical numbers,}$$

$$f_2(t) = f(\mathbf{r}_2(t)) = 5t - 1, \quad t \in [\,-2, 2\,]; \quad \text{no critical numbers.}$$

Evaluating these functions at the endpoints of their domains, we find that:

$$f_1(-2) = f_2(-2) = f(-2, 4) = -11; \qquad f_1(2) = f_2(2) = f(2, 4) = 9.$$

f takes on its absolute maximum of 9 at $(2, 4)$ and its absolute minimum of -11 at $(-2, 4)$.

11. $\boldsymbol{\nabla} f = (4 - 4x)\cos y\,\mathbf{i} + (4x - 2x^2)\sin y\,\mathbf{j} = \mathbf{0}$ at $(1, 0)$ in D: $f(1, 0) = 2$

Next we consider the boundary of D. We parametrize each side of the rectangle:

$$C_1: \ \mathbf{r}_1(t) = t\,\mathbf{j}, \quad t \in \left[\,-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi\,\right]$$

$$C_2: \ \mathbf{r}_2(t) = t\,\mathbf{i} - \tfrac{1}{4}\pi\,\mathbf{j}, \quad t \in [\,0, 2\,]$$

$$C_3: \ \mathbf{r}_3(t) = 2\,\mathbf{i} + t\,\mathbf{j}, \quad t \in \left[\,-\tfrac{1}{4}\pi, \tfrac{1}{4}\pi\,\right]$$

$$C_4: \ \mathbf{r}_4(t) = t\,\mathbf{i} + \tfrac{1}{4}\pi\,\mathbf{j}, \quad t \in [\,0, 2\,]$$

Now,

$$f_1(t) = f(\mathbf{r}_1(t)) = 0;$$

$$f_2(t) = f(\mathbf{r}_2(t)) = \frac{\sqrt{2}}{2}(4t - 2t^2), \quad t \in [\,0, 2\,]; \quad \text{critical number: } t = 1;$$

$$f_3(t) = f(\mathbf{r}_3(t)) = 0;$$

$$f_4(t) = f(\mathbf{r}_4(t)) = \frac{\sqrt{2}}{2}(4t - 2t^2), \quad t \in [\,0, 2\,]; \quad \text{critical number: } t = 1;$$

f at the vertices of the rectangle has the value 0; $\quad f_2(1) = f_4(1) = f\left(1, -\tfrac{1}{4}\pi\right) = f\left(1, \tfrac{1}{4}\pi\right) = \sqrt{2}$.

f takes on its absolute maximum of 2 at $(1, 0)$ and its absolute minimum of 0 along the lines $x = 0$ and $x = 2$.

16. $\boldsymbol{\nabla} f = (2x + 1)\,\mathbf{i} + (8y - 2)\,\mathbf{j} = \mathbf{0}$ at $\left(-\tfrac{1}{2}, \tfrac{1}{4}\right)$ in D; $\quad f\left(-\tfrac{1}{2}, \tfrac{1}{4}\right) = -\tfrac{1}{2}$

Next we consider the boundary of D. We parametrize the ellise by:

$$C: \ \mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad t \in [\,0, 2\pi\,]$$

The values of f on the boundary are given by the function

$$F(t) = f(\mathbf{r}(t)) = 4\cos^2 t + 4\sin^2 t + 2\cos t - 2\sin t = 4 + 2\cos t - 2\sin t, \quad t \in [\,0, 2\pi\,]$$

$$F'(t) = -2\sin t - 2\cos t: \quad F'(t) = 0 \quad \Longrightarrow \quad \cos t = -\sin t \quad \Longrightarrow \quad t = \tfrac{3}{4}\pi, \text{ or } \tfrac{7}{4}\pi$$

Evaluating F at the endpoints and critical numbers, we have:

$$F(0) = F(2\pi) = f(2,0) = 6;$$
$$F\left(\tfrac{3}{4}\pi\right) = f\left(-\sqrt{2}, \sqrt{2}/2\right) = 4 - 2\sqrt{2}; \quad F\left(\tfrac{7}{4}\pi\right) = f\left(\sqrt{2}, -\sqrt{2}/\right) = 4 + 2\sqrt{2}$$

f takes on its absolute maximum of $4 + 2\sqrt{2}$ at $\left(\sqrt{2}, -\sqrt{2}/2\right)$; f takes on its absolute minimum of $-\frac{1}{2}$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$.

19. Using the hint, we want to find the maximum value of $f(x,y) = 18xy - x^2y - xy^2$.

The gradient of f is:

$$\nabla D = \left(18y - 2xy - y^2\right)\, \mathbf{i} + \left(18x - x^2 - 2xy\right)\, \mathbf{j}$$

The gradient is $\mathbf{0}$ when

$$18y - 2xy - y^2 = 0 \quad \text{and} \quad 18x - x^2 - 2xy = 0$$

The solution set of this pair of equations is: $(0,0)$, $(18,0)$, $(0,18)$, $(6,6)$.

It is easy to verify that f is a maximum when $x = y = 6$. The three numbers that satisfy $x+y+z = 18$ and maximize the product xyz are: $x = 6$, $y = 6$, $z = 6$.

21. $f(x,y) = xy(1 - x - y), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$

[dom (f) is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.]

$\nabla f = (y - 2xy - y^2)\mathbf{i} + \left(x - 2xy - x^2\right)\mathbf{j} = \mathbf{0} \quad \Longrightarrow \quad x = y = \frac{1}{3}.$

(Note that $[0,0]$ is not an interior point of the domain of f.)

$f_{xx} = -2x, \quad f_{xy} = 1 - 2x - 2y, \quad f_{yy} = -2x, \quad D = (1 - 2x - 2y)^2 - 4xy.$

At $\left(\frac{1}{3}, \frac{1}{3}\right)$, $D = \frac{1}{3} > 0$ and $A < 0$ so we have a local max; the value is $1/27$.

Since $f(x,y) = 0$ at each point on the boundary of the domain, the local max of $1/27$ is also the absolute max.

24. $C = 4xy + 3(2xz + 2yz) = 4xy + 6z(x+y).$

Since $xyz = 12$, we need to minimize $C(x,y) = 4xy + \dfrac{72}{xy}(x+y), \quad x > 0, \ y > 0.$

$$\nabla C = (4y - \frac{72}{x^2})\mathbf{i} + (4x - \frac{72}{y^2})\mathbf{j} = \mathbf{0} \quad \text{at} \quad (18^{1/3}, 18^{1/3})$$

dimensions $\sqrt[3]{18} \times \sqrt[3]{18} \times \dfrac{12}{18^{2/3}}.$

33.
$$\begin{aligned} 96 &= xyz, \\ C &= 30xy + 10(2xz + 2yz) \\ &= 30xy + 20(x+y)\frac{96}{xy}. \end{aligned}$$

$$C(x,y) = 30\left[xy + \frac{64}{x} + \frac{64}{y}\right],$$

$$\nabla C = 30(y - 64x^{-2})\mathbf{i} + 30(x - 64y^{-2})\,\mathbf{j} = \mathbf{0} \quad\Longrightarrow\quad x = y = 4.$$

$$C_{xx} = 128x^{-3}, \quad C_{xy} = 1, \quad C_{yy} = 128y^{-3}.$$

When $x = y = 4$, we have $D = -3 < 0$ and $A = 2 > 0$ so the cost is minimized by making the dimensions of the crate $4 \times 4 \times 6$ meters.

35. Let x, y and z be the length, width and height of the box. The surface area is given by

$$S = 2xy + 2xz + 2yz, \quad \text{so} \quad z = \frac{S - 2xy}{2(x+y)}, \quad \text{where } S \text{ is a constant, and } x, y, z > 0.$$

Now, the volume $V = xyz$ is given by:

$$V(x,y) = xy\left[\frac{S - 2xy}{2(x+y)}\right]$$

and

$$\nabla V = y\left\{\left[\frac{S-2xy}{2(x+y)}\right] + xy\,\frac{2(x+y)(-2y) - (S-2xy)(2)}{4(x+y)^2}\right\}\mathbf{i} + \left\{x\left[\frac{S-2xy}{2(x+y)}\right] + xy\,\frac{2(x+y)(-2x) - (S-2xy)(2)}{4(x+y)^2}\right\}\mathbf{j}$$

Setting $\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} = 0$ and simplifying, we get the pair of equations

$$2S - 4x^2 - 8xy = 0$$

$$2S - 4y^2 - 8xy = 0$$

from which it follows that $x = y = \sqrt{S/6}$. From practical considerations, we conclude that V has a maximum value at $(\sqrt{S/6}, \sqrt{S/6})$. Substituting these values into the equation for z, we get $z = \sqrt{S/6}$ and so the box of maximum volume is a cube.

36. $V = xyz, \quad S = xy + 2xz + 2yz \quad\Longrightarrow\quad V(x,y) = xy\dfrac{(S - xy)}{2(x+y)}, \quad x > 0,\ y > 0,\ xy < S.$

$$\nabla V = \frac{y^2(S - x^2 - 2xy)}{2(x+y)^2}\mathbf{i} + \frac{x^2(S - y^2 - 2xy)}{2(x+y)^2}\mathbf{j}$$

$\nabla V = \mathbf{0} \quad\Longrightarrow\quad x = \sqrt{\dfrac{s}{3}}, \quad y = \sqrt{\dfrac{s}{3}}\,;$ $\qquad$ dimensions for maximum volume: $\quad \sqrt{\dfrac{s}{3}} \times \sqrt{\dfrac{s}{3}} \times \dfrac{1}{2}\sqrt{\dfrac{s}{3}}$

39. (a) Let x and y be the cross-sectional measurements of the box, and let l be its length.

Then

$$V = xyl, \quad \text{where} \quad 2x + 2y + l \le 108, \quad x, y > 0$$

To maximize V we will obviously take $2x+2y+l = 108$. Therefore, $V(x,y) = xy(108-2x-2y)$ and

$$\nabla V = [y(108 - 2x - 2y) - 2xy]\,\mathbf{i} + [x(108 - 2x - 2y) - 2xy]\,\mathbf{j}$$

Setting $\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} = 0$, we get the pair of equations

$$\frac{\partial V}{\partial x} = 108y - 4xy - 2y^2 = 0$$
$$\frac{\partial V}{\partial y} = 108x - 4xy - 2x^2 = 0$$

from which it follows that $x = y = 18 \quad \Longrightarrow \quad l = 36$.

Now, at $(18, 18)$, we have

$$A = V_{xx} = -4y = -72 < 0, \quad B = V_{xy} = 108 - 4x - 4y = -36,$$
$$C = V_{yy} = -4x = -72, \quad \text{and} \quad D = (36)^2 - (72)^2 < 0.$$

Thus, V is a maximum when $x = y = 18$ and $l = 36$.

(b) Let r be the radius of the tube and let l be its length.

Then

$$V = \pi r^2 l, \quad \text{where} \quad 2\pi r + l \le 108, \quad r > 0$$

To maximize V we take $2\pi r + l = 108$. Then $V(r) = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3$. Now

$$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2$$

Setting $\dfrac{dV}{dr} = 0$, we get

$$216\pi r - 6\pi^2 r^2 = 0 \quad \Longrightarrow \quad r = \frac{36}{\pi} \quad \Longrightarrow \quad l = 36$$

Now, at $r = 36/\pi$, we have

$$\frac{d^2V}{dr^2} = 216\pi - 12\pi^2\,\frac{36}{\pi} = -\,216\pi < 0$$

Thus, V is a maximum when $r = 36/\pi$ and $l = 36$.

SECTION 15.7

3.

$$f(x, y) = xy, \qquad g(x, y) = b^2x^2 + a^2y^2 - a^2b^2$$
$$\boldsymbol{\nabla} f = y\mathbf{i} + x\mathbf{j}, \qquad \boldsymbol{\nabla} g = 2b^2x\mathbf{i} + 2a^2y\mathbf{j}.$$
$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g \quad \Longrightarrow \quad y = 2\lambda b^2 x \;\text{ and }\; x = 2\lambda a^2 y.$$

Multiplying the first equation by a^2y and the second equation by b^2x, we get

$$a^2y^2 = 2\lambda a^2b^2xy = b^2x^2.$$

Thus, $ay = \pm bx$. From $g(x, y) = 0$ we conclude that $x = \pm\frac{1}{2}a\sqrt{2}$ and $y = \pm\frac{1}{2}b\sqrt{2}$.

Since f is continuous and the ellipse is closed and bounded, the minimum exists. It occurs at $\left(\frac{1}{2}a\sqrt{2}, -\frac{1}{2}b\sqrt{2}\right)$ and $\left(-\frac{1}{2}a\sqrt{2}, \frac{1}{2}b\sqrt{2}\right)$; the minimum is $-\frac{1}{2}ab$.

pb 8 $f(x, y, z) = xyz, \quad g(x, y, z) = x^2 + y^2 + z^2 - 1$

$\mathbf{\nabla} f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \quad \mathbf{\nabla} g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\mathbf{\nabla} f = \lambda \mathbf{\nabla} g \implies yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z \implies x^2 = y^2 = z^2.$

From $g(x,y,z) = 0$ we get $3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}},\ y = \pm\frac{1}{\sqrt{3}},\ z = \pm\frac{1}{\sqrt{3}}$

Minimum of xyz is: $-\dfrac{1}{9}\sqrt{3}$

11. Since the sphere is closed and bounded and $2x + 3y + 5z$ is continuous, the maximum exists.

$$f(x,y,z) = 2x + 3y + 5z, \qquad g(x,y,z) = x^2 + y^2 + z^2 - 19$$

$$\mathbf{\nabla} f = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{\nabla} g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

$$\mathbf{\nabla} f = \lambda \mathbf{\nabla} g \quad \implies \quad 2 = 2\lambda x, \quad 3 = 2\lambda y, \quad 5 = 2\lambda z.$$

Since $\lambda \neq 0$ here, we solve the equations for x, y and z:

$$x = \frac{1}{\lambda}, \quad y = \frac{3}{2\lambda}, \quad z = \frac{5}{2\lambda},$$

and substitute these results in $g(x,y,z) = 0$ to obtain

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{25}{4\lambda^2} - 19 = 0, \quad \frac{38}{4\lambda^2} - 19 = 0, \quad \lambda = \pm\frac{1}{2}\sqrt{2}.$$

The positive value of λ will produce positive values for x, y, z and thus the maximum for f. We get $x = \sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$, $z = \frac{5}{2}\sqrt{2}$, and $2x + 3y + 5z = 19\sqrt{2}$.

14. Maximize area $A = xy$ given that the perimeter $P = 2x + 2y$

$f(x,y) = xy, \quad g(x,y) = 2x + 2y - P$

$\mathbf{\nabla} f = y\mathbf{i} + x\mathbf{j}, \quad \mathbf{\nabla} g = 2\mathbf{i} + 2\mathbf{j}; \qquad \mathbf{\nabla} f = \lambda \mathbf{\nabla} g \implies y = 2\lambda, \quad x = 2\lambda \implies x = y.$

The rectangle of maximum area is a square.

17. It suffices to maximize and minimize the square of the distance from $(2,1,2)$ to the sphere. Clearly, these extreme values exist.

$$f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-2)^2 \qquad g(x,y,z) = x^2 + y^2 + z^2 - 1$$

$$\mathbf{\nabla} f = 2(x-2)\,\mathbf{i} + 2(y-1)\,\mathbf{j} + 2(z-2)\,\mathbf{k}, \qquad \mathbf{\nabla} g = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

$$\mathbf{\nabla} f = \lambda = \mathbf{\nabla} g \quad \implies \quad 2(x-2) = 2x\lambda, \quad 2(y-1) = 2y\lambda, \quad 2(z-2) = 2z\lambda$$

Thus,

$$x = \frac{2}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z = \frac{2}{1-\lambda}.$$

Using the fact that $x^2 + y^2 + z^2 = 1$, we have

$$\left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 1 \quad \implies \quad \lambda = -2, 4$$

At $\lambda = -2$, $\quad (x, y, z) = (2/3,\ 1/3,\ 2/3)$ $\quad$ and $\quad f(2/3, 1/3/2/3) = 4$

At $\lambda = 4$, $\quad (x, y, z) = (-2/3,\ -1/3,\ -2/3)$ $\quad$ and $\quad f(-2/3, -1/3/-2/3) = 16$

Thus, $(2/3,\ 1/3,\ 2/3)$ is the closest point and $(-2/3,\ -1/3,\ -2/3)$ is the furthest point.

20. $f(x, y, z) = xyz, \quad g(x, y, z) = x^2 + y^2 + z - 4$

$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$\nabla f = \lambda \nabla g \implies yz = 2\lambda x,\ xz = 2\lambda y,\ xy = \lambda \implies x^2 = y^2 = \dfrac{z}{2}$

From $g(x, y, z) = 0$ we get $4x^2 = 4 \implies x = 1,\ y = 1,\ z = 2.$

Maximum volume is 2

28. $f(x, y, z) = 8xyz, \quad g(x, y, z) = a^2 - x^2 - y^2 - z^2,\ x > 0,\ y > 0,\ z > 0.$

$\nabla f = 8yz\mathbf{i} + 8xz\mathbf{j} + 8xy\mathbf{k}, \quad \nabla g = -2x\,\mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$

$\nabla f = \lambda \nabla g \implies 8yz = -2\lambda x, \quad 8xz = -2\lambda y, \quad 8xy = -2\lambda z \implies x = y = z$

The rectangular box of maximum volume inscribed in the sphere is a cube.

29. (a)

$f(x, y) = (xy)^{1/2}, \qquad g(x, y) = x + y - k, (x, y \geq 0,\ k$ a nonnegative constant

$$\nabla f = \frac{y^{1/2}}{2x^{1/2}}\mathbf{i} + \frac{x^{1/2}}{2y^{1/2}}\mathbf{j} \qquad \nabla g = \mathbf{i} + \mathbf{j}.$$

$$\nabla f = \lambda \nabla g \quad \implies \quad \frac{y^{1/2}}{2x^{1/2}} = \lambda = \frac{x^{1/2}}{2y^{1/2}} \quad \implies \quad x = y = \frac{k}{2}.$$

Thus, the maximum value of f is: $f(k/2,\ k/2) = \dfrac{k}{2}$.

(b) For all $x, y \ (x, y \geq 0)$ we have

$$(xy)^{1/2} = f(x, y) \leq f(k/2,\ k/2) = \frac{k}{2} = \frac{x+y}{2}.$$

30. (a) The maximum occurs when $x = y = z = \dfrac{k}{3}$, where $(xyz)^{1/3} = \dfrac{k}{3}$.

(b) If $x + y + z = k$, then, by (a), $(xyz)^{1/3} \leq \dfrac{k}{3} = \dfrac{x+y+z}{3}$

34. $f(x, y, z) = xyz, \quad g(x, y, z) = x + y + z - 18$

$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \quad \nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\nabla f = \lambda \nabla g \quad \implies \quad yz = xz = xy \quad \implies \quad x = y = z \quad \implies \quad x = y = z = 6$

38. Let x, y, z denote the length, width and height of the box. We want to maximize the volume V of the box given that the surface area S is constant. That is:

maximize $V(x, y, z) = xyz$ $\qquad$ subject to $\qquad S(x, y, z) = 2xy + 2xz + 2yz = S$ constant

Let $g(x, y, z) = 2xy + 2xz + 2yz - S$. Then

$$\nabla V = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}, \qquad \nabla g = (2y + 2z)\,wi + (2x + 2z)\,\mathbf{j} + (2x + 2y)\,\mathbf{k}$$

$\nabla V = \lambda \nabla g$ and the side condition yield the system of equations:

$$\begin{aligned} yz &= \lambda\,(2y + 2z) \\ xz &= \lambda\,(2x + 2z) \\ xy &= \lambda\,(2x + 2y) \\ xy + 2xz + 2yz &= S. \end{aligned}$$

Multiply the first equation by x, the second by y and subtract. This gives

$$0 = 2\lambda\, z(x - y) \quad\Longrightarrow\quad x = y \quad \text{since} \;\; z = 0 \quad\Longrightarrow\quad V = 0.$$

Multiply the second equation by y, the third by z and subtract. This gives

$$0 = 2\lambda\, x(y - z) \quad\Longrightarrow\quad y = z \quad \text{since} \;\; x = 0 \quad\Longrightarrow\quad V = 0.$$

Thus the closed rectangular box of maximum volume is a cube. The cube has side length $x = \sqrt{S/6}$.

39. Let x, y, z denote the length, width and height of the box. We want to maximize the volume V of the box given that the surface area S is constant. That is:

maximize $V(x, y, z) = xyz$ subject to $S(x, y, z) = xy + 2xz + 2yz = S$ constant

Let $g(x, y, z) = xy + 2xz + 2yz - S$. Then

$$\nabla V = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}, \qquad \nabla g = (y + 2z)\,wi + (x + 2z)\,\mathbf{j} + (2x + 2y)\,\mathbf{k}$$

$\nabla V = \lambda \nabla g$ and the side condition yield the system of equations:

$$\begin{aligned} yz &= \lambda\,(y + 2z) \\ xz &= \lambda\,(x + 2z) \\ xy &= \lambda\,(2x + 2y) \\ xy + 2xz + 2yz &= S. \end{aligned}$$

Multiplying the first equation by x, the second by y and subtracting, we get

$$0 = 2\lambda\, z(x - y) \quad\Longrightarrow\quad x = y \quad \text{since} \;\; z = 0 \quad\Longrightarrow\quad V = 0.$$

Now put $y = x$ in the third equation. This gives

$$x^2 = 4\lambda\, x \quad\Longrightarrow\quad x(x - 4\lambda) = 0 \quad\Longrightarrow\quad x = 4\lambda \quad \text{since} \;\; x = 0 \quad\Longrightarrow\quad V = 0.$$

Thus, $x = y = 4\lambda$. Substituting $x = 4\lambda$ in the second equation gives $z = 2\lambda$.

Finally, substituting these values for x, y, z in the fourth equation, we get

$$48\lambda^2 = S \quad\Longrightarrow\quad \lambda^2 = \frac{S}{48} \quad\Longrightarrow\quad \lambda = \frac{1}{4}\sqrt{\frac{S}{3}}$$

To maximize the volume, take $x = y = \sqrt{\dfrac{S}{3}}$ and $z = \frac{1}{2}\sqrt{\dfrac{S}{3}}$.

SECTION 15.9

1. $\dfrac{\partial f}{\partial x} = xy^2, \quad f(x,y) = \tfrac{1}{2}x^2y^2 + \phi(y), \quad \dfrac{\partial f}{\partial y} = x^2y + \phi'(y) = x^2y.$

Thus, $\quad \phi'(y) = 0, \ \phi(y) = C, \quad$ and $\quad f(x,y) = \tfrac{1}{2}x^2y^2 + C.$

4. $\dfrac{\partial f}{\partial x} = x^2 + y \implies f(x,y) = \dfrac{x^3}{3} + xy + \phi(y); \quad \dfrac{\partial f}{\partial y} = x + \phi'(y) = y^3 + x \implies f(x,y) = \dfrac{1}{3}x^3 + \dfrac{1}{4}y^4 + xy + C$

6. $\dfrac{\partial f}{\partial x} = y^2e^x - y \implies f(x,y) = y^2e^x - xy + \phi(y);$

$\dfrac{\partial f}{\partial y} = 2ye^x - x + \phi'(y) = 2ye^x - x \implies f(x,y) = y^2e^x - xy + C$

7. $\dfrac{\partial f}{\partial x} = \cos x - y\sin x, \quad f(x,y) = \sin x + y\cos x + \phi(y), \quad \dfrac{\partial f}{\partial y} = \cos x + \phi'(y) = \cos x.$

Thus, $\quad \phi'(y) = 0, \ \phi(y) = C, \quad$ and $\quad f(x,y) = \sin x + y\cos x + C.$

12. $\dfrac{\partial f}{\partial x} = e^x + 2xy \implies f(x,y) = e^x + x^2y + \phi(y); \quad \dfrac{\partial f}{\partial y} = x^2 + \phi'(y) = x^2 + \sin y$

$\implies f(x,y) = e^x + x^2y - \cos y + C$

13. No; $\dfrac{\partial}{\partial y}\left(xe^{xy} + x^2\right) = x^2e^{xy}$ whereas $\dfrac{\partial}{\partial x}\left(ye^{xy} - 2y\right) = y^2e^{xy}$

15. $\dfrac{\partial}{\partial x} = 1 + y^2 + xy^2, \quad f(x,y) = x + xy^2 + \tfrac{1}{2}\,x^2y^2 + \phi(y), \quad \dfrac{\partial}{\partial y} = 2xy + x^2y + \phi'(y) = x^2y + y + 2xy + 1.$

Thus, $\quad \phi'(y) = y + 1, \quad \phi(y) = \tfrac{1}{2}\,y^2 + y + C \quad$ and $\quad f(x,y) = x + xy^2 + \tfrac{1}{2}\,x^2y^2 + \tfrac{1}{2}\,y^2 + y + C.$

17. $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}, \quad f(x,y) = \sqrt{x^2+y^2} + \phi(y), \quad \dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} + \phi'(y) = \dfrac{y}{\sqrt{x^2+y^2}}.$

Thus, $\quad \phi'(y) = 0, \ \phi(y) = C, \quad$ and $\quad f(x,y) = \sqrt{x^2+y^2} + C.$

25. (a) $\quad P = 2x, \ Q = z, \ R = y; \qquad \dfrac{\partial P}{\partial y} = 0 = \dfrac{\partial Q}{\partial x}, \quad \dfrac{\partial P}{\partial z} = 0 = \dfrac{\partial R}{\partial x}, \quad \dfrac{\partial Q}{\partial z} = 1 = \dfrac{\partial R}{\partial y}$

(b), (c), and (d)

$$\frac{\partial f}{\partial x} = 2x, \quad f(x,y,z) = x^2 + g(y,z).$$

$$\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = z \quad \implies \quad \frac{\partial g}{\partial y} = z.$$

Then,

$$g(y,z) = yz + h(z),$$

$$f(x,y,z) = x^2 + yz + h(z),$$

$$\frac{\partial f}{\partial z} = 0 + y + h'(z) \quad \text{and} \quad \frac{\partial f}{\partial z} = y \quad \Longrightarrow \quad h'(z) = 0.$$

Thus, $h(z) = C$ and $f(x,y,z) = x^2 + yz + C$.

26. $\dfrac{\partial f}{\partial x} = yz \implies f(x,y,z) = xyz + g(y,z);\quad \dfrac{\partial f}{\partial y} = xz + \dfrac{\partial g}{\partial y} = xz \implies f = xyz + h(z)$

$\dfrac{\partial f}{\partial z} = xy + h'(z) = xy \implies f(x,y,z) = xyz + C$

27. The function is a gradient by the test stated before Exercise 23.

Take $P = 2x + y$, $Q = 2y + x + z$, $R = y - 2z$. Then

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = 1 = \frac{\partial R}{\partial y}.$$

Next, we find f where $\nabla f = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\frac{\partial f}{\partial x} = 2x + y,$$

$$f(x,y,z) = x^2 + xy + g(y,z).$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = 2y + x + z \quad \Longrightarrow \quad \frac{\partial g}{\partial y} = 2y + z.$$

Then,

$$g(y,z) = y^2 + yz + h(z),$$

$$f(x,y,z) = x^2 + xy + y^2 + yz + h(z).$$

$$\frac{\partial f}{\partial z} = y + h'(z) = y - 2z \quad \Longrightarrow \quad h'(z) = -2z.$$

Thus, $h(z) = -z^2 + C$ and $f(x,y,z) = x^2 + xy + y^2 + yz - z^2 + C$.

29. The function is a gradient by the test stated before Exercise 25.

Take $P = y^2z^3 + 1$, $Q = 2xyz^3 + y$, $R = 3xy^2z^2 + 1$. Then

$$\frac{\partial P}{\partial y} = 2yz^3 = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = 3y^2z^2 = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = 6xyz^2 = \frac{\partial R}{\partial y}.$$

Next, we find f where $\nabla f = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\frac{\partial f}{\partial x} = y^2 z^3 + 1,$$

$$f(x, y, z) = xy^2 z^3 + x + g(y, z).$$

$$\frac{\partial f}{\partial y} = 2xyz^3 \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = 2xyz^3 + y \quad \Longrightarrow \quad \frac{\partial g}{\partial y} = y.$$

Then,

$$g(y, z) = \tfrac{1}{2}\, y^2 + h(z),$$

$$f(x, y, z) = xy^2 z^3 + x + \tfrac{1}{2}\, y^2 + h(z).$$

$$\frac{\partial f}{\partial z} = 3xy^2 z^2 + h'(z) = 3xy^2 z^2 + 1 \quad \Longrightarrow \quad h'(z) = 1.$$

Thus, $\quad h(z) = z + C \quad$ and $\quad f(x, y, z) = xy^2 z^3 + x + \tfrac{1}{2}\, y^2 + z + C.$