# CHAPTER 16

## SECTION 16.2

1. $$\sum_{i=1}^{3}\sum_{j=1}^{3} 2^{i-1}3^{j+1} = \left(\sum_{i=1}^{3} 2^{i-1}\right)\left(\sum_{j=1}^{3} 3^{j+1}\right) = (1+2+4)(9+27+81) = 819$$

3. $$\sum_{i=1}^{4}\sum_{j=1}^{3}(i^2+3i)(j-2) = \left[\sum_{i=1}^{4}(i^2+31)\right]\left[\sum_{j=1}^{3}(j-2)\right] = (4+10+18+28)(-1+0+1) = 0$$

4. $$\frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\frac{4}{2}+\frac{4}{3}+\frac{4}{4}+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{6}{2}+\frac{6}{3}+\frac{6}{4}+\frac{6}{5}+\frac{6}{6}+\frac{6}{7} = 19\frac{4}{35}.$$

5. $$\sum_{i=1}^{m}\Delta x_i = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = (x_1-x_0)+(x_2-x_1)+\cdots+(x_n-x_{n-1})$$
$$= x_n - x_0 = a_2 - a_1$$

6. $$(y-y_0)+(y_2-y_1)+\cdots+(y_n-y_{n-1}) = y_n - y_0 = b_2 - b_1$$

10. $$\sum_{j=1}^{n}\frac{1}{2}\left(y_j{}^2 + y_jy_{j-1} + y_{j-1}{}^2\right)\Delta y_j = \frac{1}{2}\sum_{j=1}^{n}\left(y_j{}^3 - y_{j-1}{}^3\right) = \frac{1}{2}\left(b_2{}^3 - b_1{}^3\right)$$

11. $$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i+x_{i-1})\,\Delta x_i\,\Delta y_j = \left(\sum_{i=1}^{m}(x_i+x_{i-1})\,\Delta x_i\right)\left(\sum_{j=1}^{n}\Delta y_j\right)$$

(Exercise 9)

$$= \left(a_2{}^2 - a_1{}^2\right)(b_2-b_1)$$

14. $$\sum_{i=1}^{m}\sum_{j=1}^{n}(3\Delta x_i - 2\Delta y_j) = 3\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta x_i - 2\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta y_j = 3n(a_2-a_1) - 2m(b_2-b_1).$$

## SECTION 16.3

3. $$\int_0^1\int_0^3 xy^2\,dy\,dx = \int_0^1 x\left[\frac{1}{3}y^3\right]_0^3 dx = \int_0^1 9x\,dx = \frac{9}{2}$$

5. $$\int_0^1\int_0^x xy^3\,dy\,dx = \int_0^1 x\left[\frac{1}{4}y^4\right]_0^x dx = \int_0^1 \frac{1}{4}x^5\,dx = \frac{1}{24}$$

7. $$\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dy\,dx = \int_0^{\pi/2}\left[-\cos(x+y)\right]_0^{\pi/2} dx = \int_0^{\pi/2}\left[\cos x - \cos\left(x+\frac{\pi}{2}\right)\right] dx = 2$$

10. $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x+3y^3)\,dx\,dy = \int_{-1}^{1}6y^3\sqrt{1-y^2}\,dy = 0 \quad \text{(integrand is odd)}$$

11. $\displaystyle\int_0^1 \int_{y^2}^{y} \sqrt{xy}\, dx\, dy = \int_0^1 \sqrt{y}\left[\frac{2}{3}x^{3/2}\right]_{y^2}^{y} dy = \int_0^1 \frac{2}{3}\left(y^2 - y^{7/2}\right) dy = \frac{2}{27}$

14. $\displaystyle I = \int_0^1 \int_{x^3}^{x^2} (x^4 + y^2)\, dy dx = \int_0^1 \left[x^4 y + \frac{y^3}{3}\right]_{x^3}^{x^2} dx = \int_0^1 \left(\frac{4x^6}{3} - x^7 - \frac{x^9}{3}\right) dx$

$\displaystyle = \left[\frac{4x^7}{21} - \frac{x^8}{8} - \frac{x^{10}}{30}\right]_0^1 = \frac{9}{280}$

16. $\displaystyle\int_0^1 \int_0^{2y} e^{-y^2/2}\, dx dy = \int_0^1 2y e^{-y^2/2}\, dy = \left[-2e^{-y^2/2}\right]_0^1 = 2\left(1 - \frac{1}{\sqrt{e}}\right)$

17. $\displaystyle\int_0^2 \int_0^{x/2} e^{x^2} dy\, dx = \int_0^2 \frac{1}{2} x e^{x^2} dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}\left(e^4 - 1\right)$

19. $\displaystyle\int_0^1 \int_{y^{1/2}}^{y^{1/4}} f(x,y)\, dx\, dy$

22. $\displaystyle\int_{\sqrt[3]{1/2}}^{1/2} \int_{1/2}^{y^{1/3}} f(x,y)\, dx dy + \int_{1/2}^{1} \int_{y}^{y^{1/3}} f(x,y)\, dx dy$

23. $\displaystyle\int_1^2 \int_1^y f(x,y)\, dx\, dy + \int_2^4 \int_{y/2}^{y} f(x,y)\, dx\, dy + \int_4^8 \int_{y/2}^{4} f(x,y)\, dx\, dy$

25. $\displaystyle\int_{-2}^4 \int_{1/4x^2}^{\frac{1}{2}x+2} dy\, dx = \int_{-2}^4 \left[\frac{1}{2}x + 2 - \frac{1}{4}x^2\right] dx = 9$

29. $\displaystyle\int_0^1 \int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx\, dy = \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy = \left[-\frac{2}{3}\cos\left(\frac{y^3+1}{2}\right)\right]_0^1 = \frac{2}{3}\left(\cos\frac{1}{2} - \cos 1\right)$

31. $\displaystyle\int_0^{\ln 2} \int_{e^x}^{2} e^{-x}\, dy\, dx = \int_0^{\ln 2} e^{-x}\left(2 - e^x\right) dx = \left[-2e^{-x} - x\right]_0^{\ln 2} = 1 - \ln 2$

32. $\displaystyle\int_0^1 \int_0^{\sqrt{y}} \frac{x^3}{\sqrt{x^4+y^2}}\, dx\, dy = \int_0^1 \frac{1}{2}(\sqrt{2}-1) y\, dy = \frac{1}{4}(\sqrt{2}-1)$

35. $\displaystyle\int_0^2 \int_0^{3-\frac{3}{2}x} \left(4 - 2x - \frac{4}{3}y\right) dy\, dx = \int_0^3 \int_0^{2-\frac{2}{3}y} \left(4 - 2x - \frac{4}{3}y\right) dx\, dy = 4$

37. $\displaystyle\int_0^2 \int_0^{1-\frac{1}{2}x} x^3 y\, dy\, dx = \int_0^2 \int_0^{2-2y} x^3 y\, dx\, dy = \frac{2}{15}$

38. $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx = 2\int_{-1}^1 \left[x^2\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}\right] dx = \frac{\pi}{2}$

41. $\displaystyle\int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx = \int_0^1 \left(2x^2 - \frac{4}{3}x^3 - x + \frac{1}{3}\right) dx = \frac{1}{6}$

47. $\displaystyle\int_0^1 \int_y^1 e^{y/x} dx\, dy = \int_0^1 \int_0^x e^{y/x} dy\, dx = \int_0^1 \left[x e^{y/x}\right]_0^x dx = \int_0^1 x(e-1)\, dx = \frac{1}{2}(e-1)$

49. $\displaystyle\int_0^1 \int_x^1 x^2 e^{y^4} dy\, dx = \int_0^1 \int_0^y x^2 e^{y^4} dx\, dy = \int_0^1 \left[\frac{1}{3}x^3 e^{y^4}\right]_0^y dy = \frac{1}{3}\int_0^1 y^3 e^{y^4} dy = \frac{1}{12}(e-1)$

SECTION 16.4

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}\sin^2\theta\cos\theta\,d\theta = \left[\frac{1}{6}\sin^3\theta\right]_0^{\pi/2} = \frac{1}{6}$

4. $\displaystyle\int_{-\pi/3}^{2\pi/3}\int_0^{2\cos\theta} r\sin\theta\,dr\,d\theta = \int_{-\pi/3}^{2\pi/3} 2\cos^2\theta\sin\theta\,d\theta = \frac{1}{6}$

5. (a) $\Gamma:\ 0\le\theta\le 2\pi,\quad 0\le r\le 1$

$$\iint_\Gamma \left(\cos r^2\right) r\,dr d\theta = \int_0^{2\pi}\int_0^1 \left(\cos r^2\right) r\,dr\,d\theta = 2\pi\int_0^1 r\cos r^2\,dr = \pi\sin 1$$

(b) $\Gamma:\ 0\le\theta\le 2\pi,\quad 1\le r\le 2$

$$\iint_\Gamma \left(\cos r^2\right) r\,dr d\theta = \int_0^{2\pi}\int_1^2 \left(\cos r^2\right) r\,dr\,d\theta = 2\pi\int_1^2 r\cos r^2\,dr = \pi(\sin 4-\sin 1)$$

7. (a) $\Gamma:\ 0\le\theta\le \pi/2,\quad 0\le r\le 1$

$$\iint_\Gamma (r\cos\theta + r\sin\theta) r\,dr d\theta = \int_0^{\pi/2}\int_0^1 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \left(\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_0^1 r^2\,dr\right) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

(b) $\Gamma:\ 0\le\theta\le \pi/2,\quad 1\le r\le 2$

$$\iint_\Gamma (r\cos\theta + r\sin\theta) r\,dr d\theta = \int_0^{\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \left(\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = 2\left(\frac{7}{3}\right) = \frac{14}{3}$$

8. $\Gamma:\ 0\le\theta\le\frac{\pi}{3},\quad 0\le r\le\frac{1}{\cos\theta}$

$$\iint_\Gamma \sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/3}\int_0^{1/\cos\theta} r\cdot r\,dr\,d\theta = \int_0^{\pi/3}\frac{d\theta}{3\cos^3\theta} = \frac{1}{3}\int_0^{\pi/3}\sec^3\theta\,d\theta$$

$$= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/3} = \frac{1}{3}\sqrt{3} + \frac{1}{6}\ln(2+\sqrt{3})$$

9. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\,dr\,d\theta = \frac{1}{3}\pi$

11. $\displaystyle\int_{1/2}^1\int_0^{\sqrt{1-x^2}} dy\,dx = \int_0^{\pi/3}\int_{\frac{1}{2}\sec\theta}^1 r\,dr\,d\theta = \int_0^{\pi/3}\left(\frac{1}{2}-\frac{1}{8}\sec^2\theta\right)d\theta = \frac{1}{6}\pi - \frac{\sqrt{3}}{8}$

13. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} \sin\sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 \sin(r)\,r\,dr\,d\theta = \int_0^{\pi/2}(\sin 1-\cos 1)\,d\theta = \frac{\pi}{2}(\sin 1-\cos 1)$

14. $\displaystyle\int_0^{2\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta = 2\pi\int_0^1 re^{-r^2}\,dr = \pi(1-\frac{1}{e})$

19. First we find the points of intersection:

$r = 4\cos\theta = 2 \quad\Longrightarrow\quad \cos\theta = \dfrac{1}{2}$

$\Longrightarrow \quad \theta = \pm\dfrac{\pi}{3}.$

$$A = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}(8-\cos^2\theta-2)\,d\theta = \int_{-\pi/3}^{\pi/3}(2+4\cos 2\theta)\,d\theta = \frac{4\pi}{3}+2\sqrt{3}$$

20. $\displaystyle A = \int_0^{2\pi}\int_0^{1+2\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}(1+2\cos\theta)^2\,d\theta = 3\pi$

23.
$$\int_0^{2\pi}\int_0^b \left(r^2\sin\theta + br\right)dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{3}r^3\sin\theta+\frac{b}{2}r^2\right]_0^b d\theta$$
$$= b^3\int_0^{2\pi}\left(\frac{1}{3}\sin\theta+\frac{1}{2}\right)d\theta = b^3\pi$$

24. $\displaystyle V = \int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = 2\pi\int_0^1 (r-r^3)\,dr = \frac{\pi}{2}$

27.
$$\int_0^{2\pi}\int_0^1 r\sqrt{4-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}\right]_0^1 d\theta$$
$$= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = \frac{2}{3}(8-3\sqrt{3})\pi$$

28. $\displaystyle V = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(1-r^2)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{\cos^2\theta}{2}-\frac{\cos^4\theta}{4}\right)d\theta = \frac{5\pi}{32}$

29.
$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} 2r^2\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{2}{3}r^3\cos\theta\right]_0^{2\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\frac{16}{3}\cos^4\theta\,d\theta = \frac{32}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{32}{3}\left(\frac{3}{16}\pi\right) = 2\pi$$

Ex. 46, Sect. 8.3

30. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{8a^3}{3}\cos^3\theta d\theta = \frac{32}{9}a^3$

SECTION 16.5

1. $M = \int_{-1}^{1}\int_0^1 x^2\,dy\,dx = \frac{2}{3}$

$$x_M\,M = \int_{-1}^{1}\int_0^1 x^3\,dy\,dx = 0 \quad \Longrightarrow \quad x_M = 0$$

$$y_M\,M = \int_{-1}^{1}\int_0^1 x^2y\,dy\,dx = \int_{-1}^{1}\frac{1}{2}x^2\,dx = \frac{1}{3} \quad \Longrightarrow \quad y_M = \frac{1/3}{1/2} = \frac{1}{2}$$

3. $M = \int_0^1\int_{x^2}^1 xy\,dy\,dx = \frac{1}{2}\int_0^1 (x - x^5)\,dx = \frac{1}{6}$

$$x_M\,M = \int_0^1\int_{x^2}^1 x^2y\,dy\,dx = \frac{1}{2}\int_0^1 (x^2 - x^6)\,dx = \frac{2}{21} \quad \Longrightarrow \quad x_M = \frac{2/21}{1/6} = \frac{4}{7}$$

$$y_M\,M = \int_0^1\int_{x^2}^1 xy^2\,dy\,dx = \frac{1}{3}\int_0^1 (x - x^7)\,dx = \frac{1}{8} \quad \Longrightarrow \quad y_M = \frac{1/8}{1/6} = \frac{3}{4}$$

6. $M = \int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a \frac{x}{2}(a^2 - x^2)\,dx = \frac{a^4}{8}$

$$x_M M = \int_0^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx = \int_0^a \frac{x^2}{2}(a^2 - x^2)\,dx = \frac{a^5}{15} \Longrightarrow x_M = \frac{8}{15}a$$

$$y_M M = \int_0^a\int_0^{\sqrt{a^2-x^2}} xy^2\,dy\,dx = \int_0^a \frac{x}{3}(a^2 - x^2)^{3/2}\,dx = \frac{a^5}{15} \Longrightarrow y_M = \frac{8}{15}a$$

7. $M = \int_0^1\int_{2x}^{3x} xy\,dy\,dx = \frac{5}{2}\int_0^1 x^3\,dx = \frac{5}{8}$

$$x_M\,M = \int_0^1\int_{2x}^{3x} x^2y\,dy\,dx = \frac{5}{2}\int_0^1 x^4\,dx = \frac{1}{2} \Longrightarrow \quad x_M = \frac{1/2}{5/8} = \frac{4}{5}$$

$$y_M\,M = \int_0^1\int_{2x}^{3x} xy^2\,dy\,dx = \frac{19}{3}\int_0^1 x^4\,dx = \frac{19}{15} \quad \Longrightarrow \quad y_M = \frac{19/15}{5/8} = \frac{152}{75}$$

29. $M = \iint_\Omega k\left(R - \sqrt{x^2 + y^2}\right)dxdy = k\int_0^\pi\int_0^R \left(Rr - r^2\right)dr\,d\theta = \frac{1}{6}k\pi R^3$

$x_M = 0$ by symmetry

$$y_M M = \iint_\Omega y\left[k\left(R - \sqrt{x^2 + y^2}\right)\right]dxdy = k\int_0^\pi\int_0^R \left(Rr^2 - r^3\right)\sin\theta\,dr\,d\theta = \frac{1}{6}kR^4$$

$y_M = R/\pi$

31. Place $P$ at the origin.

$$M = \iint_{\Omega} k\sqrt{x^2 + y^2}\, dxdy$$

$$= k\int_0^{\pi}\int_0^{2R\sin\theta} r^2\, dr\, d\theta = \frac{32}{9}kR^3$$

$$x_M = 0 \quad \text{by symmetry}$$

$$y_M M = \iint_{\Omega} y\left(k\sqrt{x^2 + y^2}\right) dxdy = k\int_0^{\pi}\int_0^{2R\sin\theta} r^3 \sin\theta\, dr\, d\theta = \frac{64}{15}kR^4$$

$$y_M = 6R/5$$

Answer: the center of mass lies on the diameter through $P$ at a distance $6R/5$ from $P$.

## SECTION 16.6

1. They are equal; they both give the volume of $T$.

2. (a) $L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q} x_{i-1}y_{j-1}z_{k-1}\Delta x_i \Delta y_j \Delta z_k, \quad U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q} x_i y_j z_k \Delta x_i \Delta y_j \Delta z_k$

   (b) $x_{i-1}y_{j-1}z_{k-1} \le \left(\frac{x_i + x_{i-1}}{2}\right)\left(\frac{y_j + y_{j-1}}{2}\right)\left(\frac{z_k + z_{k-1}}{2}\right) \le x_i y_j z_k$

$$x_{i-1}y_{j-1}z_{k-1}\Delta x_i \Delta y_j \Delta z_k \le \frac{1}{8}\left(x_i{}^2 - x_{i-1}{}^2\right)\left(y_j{}^2 - y_{j-1}{}^2\right)\left(z_k{}^2 - z_{k-1}{}^2\right) \le x_i y_j z_k \Delta x_i \Delta y_j \Delta z_k$$

$$L_f(P) \le \frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\left(x_i{}^2 - x_{i-1}{}^2\right)\left(y_j{}^2 - y_{j-1}{}^2\right)\left(z_k{}^2 - z_{k-1}{}^2\right) \le U_f(P).$$

The middle term can be written

$$\frac{1}{8}\left(\sum_{i=1}^{m} x_i{}^2 - x_{i-1}{}^2\right)\left(\sum_{j=1}^{n} y_j{}^2 - y_{j-1}{}^2\right)\left(\sum_{k=1}^{q} z_k{}^2 - z_{k-1}{}^2\right) = \frac{1}{8}(1)(1)(1) = \frac{1}{8}.$$

Therefore $I = \frac{1}{8}$.

3. $$\int_0^1\int_1^{2y}\int_0^x (x+2z)\,dz\,dx\,dy = \int_0^1\int_1^{2y}\left[xz+z^2\right]_0^x\,dx\,dy = \int_0^1\int_1^{2y} 2x^2\,dx\,dy$$
$$= \int_0^1\left[\frac{2}{3}x^3\right]_1^{2y}dy = \int_0^1\left(\frac{16}{3}y^3-\frac{2}{3}\right)dy = \frac{2}{3}$$

4. $$\int_0^1\int_{1-x}^{1+x}\int_0^{xy} 4z\,dz\,dy\,dx = \int_0^1\int_{1-x}^{1+x} 2x^2y^2\,dy\,dx = \int_0^1\frac{2x^2}{3}\left[(1+x)^3-(1-x)^3\right]\,dx = \frac{11}{9}$$

5. $$\int_0^2\int_{-1}^1\int_1^3 (z-xy)\,dz\,dy\,dx = \int_0^2\int_{-1}^1\left[\frac{1}{2}z^2-xyz\right]_1^3 dy\,dx$$
$$= \int_0^2\int_{-1}^1 (4-2xy)\,dy\,dx = \int_0^2\left[2y-xy^2\right]_{-1}^1\,dx = \int_0^2 8\,dy = 16$$

7. $$\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-x^2}} x\cos z\,dy\,dx\,dz = \int_0^{\pi/2}\int_0^1 [xy\cos z]_0^{\sqrt{1-x^2}}\,dx\,dz$$
$$= \int_0^{\pi/2}\int_0^1 x\sqrt{1-x^2}\cos z\,dx\,dz = \int_0^{\pi/2}\left[-\frac{1}{3}(1-x^2)^{3/2}\cos z\right]_0^1 dz = \frac{1}{3}\int_0^{\pi/2}\cos z\,dz = \frac{1}{3}$$

8. $$\int_{-1}^2\int_1^{y+2}\int_e^{e^2}\frac{x+y}{z}\,dz\,dx\,dy = \int_{-1}^2\int_1^{y+2}(x+y)\,dx\,dy = \int_{-1}^2\left[\frac{(y+2)^2-1}{2}+(y+1)y\right]dy = \frac{27}{2}$$

9. $$\int_1^2\int_y^{y^2}\int_0^{\ln x} ye^z\,dz\,dx\,dy = \int_1^2\int_y^{y^2}[ye^z]_0^{\ln x}\,dx\,dy$$
$$= \int_1^2\int_y^{y^2} y(x-1)\,dx\,dy = \int_1^2\left[\frac{1}{2}x^2y-xy\right]_y^{y^2}dy = \int_1^2\left(\frac{1}{2}y^5-\frac{3}{2}y^3+y^2\right)dy = \frac{47}{24}$$

13. $$\left(\int_0^1 x^2\,dx\right)\left(\int_0^2 y^2\,dy\right)\left(\int_0^3 z^2\,dz\right) = \left(\frac{1}{3}\right)\left(\frac{8}{3}\right)\left(\frac{27}{3}\right) = 8$$

14. $$M = \iiint_\Pi kxyz\,dx\,dy\,dz = k\left(\int_0^a x\,dx\right)\left(\int_0^b y\,dy\right)\left(\int_0^c z\,dz\right) = \frac{1}{8}ka^2b^2c^2$$

15. $$x_M M = \iiint_\Pi kx^2yz\,dxdydz = k\left(\int_0^a x^2\,dx\right)\left(\int_0^b y\,dy\right)\left(\int_0^c z\,dz\right)$$
$$= k\left(\tfrac{1}{3}a^3\right)\left(\tfrac{1}{2}b^2\right)\left(\tfrac{1}{2}c^2\right) = \tfrac{1}{12}ka^3b^2c^2.$$

By Exercise 14, $M = \frac{1}{8} ka^2 b^2 c^2$. Therefore $\overline{x} = \frac{2}{3} a$. Similarly, $\overline{y} = \frac{2}{3} b$ and $\overline{z} = \frac{2}{3} c$.

23. $\displaystyle\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \int_{-2x-3y-10}^{1-y^2} dz\,dy\,dx$

24. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{1-x^2/2}}^{\sqrt{1-x^2/2}} \int_{2+y^2}^{4-x^2-y^2} dz\,dy\,dx$

27. $\displaystyle\iiint_T \left(x^2 z + y\right) dx\,dy\,dz = \int_0^2 \int_1^3 \int_0^1 \left(x^2 z + y\right) dx\,dy\,dz = \int_0^2 \int_1^3 \left[\frac{1}{3} x^3 z + xy\right]_0^1 dy\,dz$

$\displaystyle= \int_0^2 \int_1^3 \left(\frac{1}{3} z + y\right) dy\,dz = \int_0^2 \left[\frac{1}{3} zy + \frac{1}{2} y^2\right]_1^3 dz = \int_0^2 \left(\frac{2}{3} z + 4\right) dz = \frac{28}{3}$

28. $\displaystyle\int_0^1 \int_0^y \int_0^{x+y} 2ye^x\,dz\,dx\,dy = \int_0^1 \int_0^y 2y(x+y)e^x\,dx\,dy = \int_0^1 (4y^2 e^y - 2ye^y + 2y - 2y^2)\,dy = 4e - \frac{29}{3}$

35. $\displaystyle V = \int_{-1}^2 \int_0^3 \int_{2-x}^{4-x^2} dz\,dy\,dx = \tfrac{27}{2}; \quad (\overline{x}, \overline{y}, \overline{z}) = \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{12}{5}\right)$

44. $\displaystyle\int_0^2 \int_0^2 \int_{2-y}^{4-y^2} x^2 y^2\,dz\,dy\,dx = \frac{352}{45}$

45. $\displaystyle V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$

polar coordinates

$\displaystyle= 8\int_0^{\pi/2} \int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = -4\int_0^{\pi/2} \left[\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^a d\theta = \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4}{3}\pi a^3$

48. using polar coordinates $\displaystyle V = 2\int_0^{2\pi} \int_0^1 (r - r^3)\,dr\,d\theta = \pi$

51. (a) $\displaystyle V = \int_0^6 \int_{z/2}^3 \int_x^{6-x} dy\,dx\,dz$ (b) $\displaystyle V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy\,dz\,dx$

(c) $\displaystyle V = \int_0^6 \int_{z/2}^3 \int_{z/2}^y dx\,dy\,dz + \int_0^6 \int_3^{(12-z)/2} \int_{z/2}^{6-y} dx\,dy\,dz$

52. (a) $\displaystyle V = \iint_{\Omega_{xy}} 2\sqrt{4-y}\,dx\,dy$ (b) $\displaystyle V = \iint_{\Omega_{xy}} \left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz\right) dx\,dy$

(c) $\displaystyle V = \int_{-4}^4 \int_{|x|}^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz\,dy\,dx$ (d) $\displaystyle V = \int_0^4 \int_{-y}^y \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz\,dx\,dy$

53. (a) $\displaystyle V = \iint_{\Omega_{yz}} 2y\,dy\,dz$ (b) $\displaystyle V = \iint_{\Omega_{yz}} \left(\int_{-y}^y dx\right) dy\,dz$

(c) $V = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-y}^{y} dx\, dz\, dy$ (d) $V = \int_{-2}^{2} \int_0^{4-z^2} \int_{-y}^{y} dx\, dy\, dz$

SECTION 16.8

8. $\int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta = \frac{\pi}{12}$

9. $\int_0^{2\pi} \int_0^2 \int_0^{r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi$

10. $\int_0^3 \int_0^{2\pi} \int_r^3 r\, dz\, d\theta\, dr = 9\pi$

11.
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{4-(x^2+y^2)}} = \int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$
$$= \int_0^{\pi/2} \int_0^1 r\sqrt{4-r^2}\, dr\, d\theta$$
$$= \int_0^{\pi/2} \left(\frac{8}{3} - \sqrt{3}\right) d\theta = \frac{1}{6}\left(8 - 3\sqrt{3}\right)\pi$$

12. $\int_0^{\pi} \int_0^1 \int_r^1 z^3 r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^1 \frac{1}{4}(1-r^4)\, r\, dr\, d\theta = \frac{\pi}{12}$

15. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^2 \sin(x^2 + y^2)\, dz\, dy\, dx = \int_0^{\pi/2} \int_0^1 \int_0^2 \sin(r^2) r\, dz\, dr\, d\theta = 2\int_0^{\pi/2} \int_0^1 r \sin(r^2)\, dr\, d\theta$

16. $\int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^2 dz\, dr\, d\theta = 4\pi \int_0^1 (1-r^2) r^2\, dr = \frac{8\pi}{15}$

17.
$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \int_0^r r\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\theta\, d\theta = \frac{32}{9} a^3$$

19.
$$V = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \int_0^{a-r} r\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} r(a-r)\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} a^3 \left(\frac{1}{2}\cos^2\theta - \frac{1}{3}\cos^3\theta\right) d\theta = \frac{1}{36} a^3 (9\pi - 16)$$

22. $V = \int_0^{2\pi}\int_0^3\int_{r+1}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta = \frac{41}{3}\pi$

23. $V = \int_0^{2\pi}\int_0^{1/2}\int_{r\sqrt{3}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1/2}\left(r\sqrt{1-r^2} - r^2\sqrt{3}\right)dr\,d\theta = \frac{1}{3}\pi\left(2-\sqrt{3}\right)$

25. $V = \int_0^{2\pi}\int_1^3\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^3 r\sqrt{9-r^2}\,dr\,d\theta = \frac{32}{3}\pi\sqrt{2}$

35. $V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \frac{1}{2}\pi$

36. $M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kzr\,dz\,dr\,d\theta = \frac{1}{6}\pi k$

37. $M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} k\left(r^2+z^2\right)r\,dz\,dr\,d\theta = \frac{1}{4}k\pi$

## SECTION 16.9

3. $\left(\frac{3}{4},\ \frac{3}{4}\sqrt{3},\ \frac{3}{2}\sqrt{3}\right)$

4. $\left(2\sqrt{10},\ \frac{2}{3}\pi,\ \cos^{-1}[\frac{3}{10}\sqrt{10}]\right)$

7. $x = \rho\sin\varphi\cos\theta = 3\sin 0\cos(\pi/2) = 0$ $\quad y = \rho\sin\varphi\sin\theta = 3\sin 0\sin(\pi/2) = 0$

$z = \rho\cos\varphi = 3\cos 0 = 3$ $\quad (x,y,z) = (0,0,3)$

8. $(1,\ \pi/4,\ \pi/4)$

15. Sphere of radius 2 centered at the origin:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}$$

16. That part of the sphere of radius 1 that lies in the first quadrant between the $x,z$-plane and the plane $y = x$

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{pi}{12}$$

18. A cone of rdius 1 and height 1; $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \frac{1}{3}\pi$

19.
$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$
$$= \frac{2}{3}\sqrt{2}\int_0^{\pi/4}\int_0^{\pi/2}\sin\varphi\,d\theta\,d\varphi$$
$$= \frac{\sqrt{2}}{3}\pi\int_0^{\pi/4}\sin\varphi\,d\varphi = \frac{\sqrt{2}}{6}\pi\,(2-\sqrt{2})$$

20. $\displaystyle\int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\varphi \, d\rho \, d\theta \, d\varphi = \frac{16\pi}{5}\int_0^{\pi/4} \sin\varphi \, d\varphi = \frac{8\pi}{5}(2-\sqrt{2})$

21. $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+x^2}\, dz\, dx\, dy$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\,\cos\varphi\cdot\rho\cdot\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\frac{1}{2}\sin 2\varphi\, d\varphi\int_0^{\pi/2} d\theta\int_0^3 \rho^4\, d\rho = \left[-\frac{1}{4}\cos 2\varphi\right]_0^{\pi/2}\left(\frac{\pi}{2}\right)\left[\frac{1}{5}\rho^5\right]_0^3$$

$$= \frac{243\pi}{20}$$

23. $\displaystyle V = \int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{4}{3}\pi R^3$

25. $\displaystyle V = \int_0^{\alpha}\int_0^{\pi}\int_0^R \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{2}{3}\alpha R^3$

26. $\displaystyle M = \int_0^{2\pi}\int_0^{\pi}\int_0^R k(R-\rho)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{1}{3}k\pi R^4$

34. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 e^{\rho^3}\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{1}{3}\pi(e-1)\left(2-\sqrt{2}\right)$