CHAPTER 17

SECTION 17.1

1. (a) $\mathbf{h}(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = u\,\mathbf{i} + u^2\,\mathbf{j}, \quad u \in [0,1]$

$$x(u) = u, \quad y(u) = u^2; \quad x'(u) = 1, \quad y'(u) = 2u$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = u^2(1) + u(2u) = 3u^2$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 3u^2\,du = 1$$

(b) $h(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = u^3\,\mathbf{i} - 2u\,\mathbf{j}, \quad u \in [0,1]$

$$x(u) = u^3, \quad y(u) = -2u; \quad x'(u) = 3u^2, \quad y'(u) = -2$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = (-2u)(3u^2) + u^3(-2) = -8u^3$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 -8u^3\,du = -2$$

3. $h(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = \cos u\,\mathbf{i} - \sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$

$$x(u) = \cos u, \quad y(u) = -\sin u; \quad x'(u) = -\sin u, \quad y'(u) = -\cos u$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = \sin^2 u - \cos^2 u$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} (\sin^2 u - \cos^2 u)\,du = 0$$

4. (a) $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^1 (e^{-u}\mathbf{i} + 2\mathbf{j}) \cdot (e^u\mathbf{i} - e^{-u}\mathbf{j})\,du = \int_0^1 (1 - 2e^{-u})\,du = 2e^{-1} - 1$

(b) $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^2 2\mathbf{j} \cdot (1-u)\mathbf{i}\,du = \int_0^2 0\,du = 0$

5. (a) $\mathbf{r}(u) = (2-u)\,\mathbf{i} + (3-u)\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 (-5 + 5u - u^2)\,du = -\frac{17}{6}$$

(b) $\mathbf{r}(u) = (1+u)\,\mathbf{i} + (2+u)\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 (1 + 3u + u^2)\,du = \frac{17}{6}$$

7. $C = C_1 \cup C_2 \cup C_3$ where,

$$C_1 : \mathbf{r}(u) = (1-u)(-2\,\mathbf{i}) + u(2\,\mathbf{i}) = (4u-2)\,\mathbf{i}, \quad u \in [0,1]$$

$$C_2 : \mathbf{r}(u) = (1-u)(2\,\mathbf{i}) + u(2\,\mathbf{j}) = (2-2u)\,\mathbf{i} + 2u\,\mathbf{j}, \quad u \in [0,1]$$

$C_3\colon \mathbf{r}(u) = (1-u)(2\,\mathbf{j}) + u(-2\,\mathbf{i}) = -2u\,\mathbf{i} + (2-2u)\,\mathbf{j}, \quad u \in [\,0,1\,]$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-4) + (-4) = -8$$

9. $C_1\colon \mathbf{r}(u) = (-1+2u)\,\mathbf{i}, \quad u \in [\,0,1\,] \qquad C_2\colon \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}, \quad u \in [\,0,\pi\,]$

$$\int_C = \int_{C_1} + \int_{C_2} = 0 + (-\pi) = -\pi$$

10. Bottom: $\mathbf{r}(u) = u\mathbf{i}$; $\displaystyle\int_0^1 u^3\mathbf{j}\cdot\mathbf{i}\,du = \int_0^1 0\,du = 0$

Right side: $\mathbf{r}(u) = \mathbf{i} + u\mathbf{j}$; $\displaystyle\int_0^1 [3u\mathbf{i} + (1+2u)\mathbf{j}]\cdot\mathbf{j}\,du = \int_0^1 (1+2u)\,du = 2$

Top: $\mathbf{r}(u) = (1-u)\mathbf{i} + \mathbf{j}$; $\displaystyle\int_0^1 3(1-u)^2\mathbf{i}\cdot(-\mathbf{i})\,du = \int_0^1 -3(1-u)^2\,du = -1$

Left: $\mathbf{r}(u) = (1-u)\mathbf{j}$; $\displaystyle\int_0^1 2(1-u)\mathbf{j}\cdot(-\mathbf{j})\,du = \int_0^1 -2(1-u)\,du = -1$

$$\int_C \mathbf{h}\cdot d\mathbf{r} = \text{sum of the above} = 0$$

11. (a) $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j} + u\mathbf{k}, \quad u \in [\,0,1\,]$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 3u^2\,du = 1$$

(b) $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (2u^3 + u^5 + 3u^6)\,du = \frac{23}{21}$

13. (a) $\mathbf{r}(u) = 2u\,\mathbf{i} + 3u\,\mathbf{j} - u\,\mathbf{k}, \quad u \in [\,0,1\,]$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (2\,\cos 2u + 3\,\sin 3u + 3u^2)\,du = \left[\sin 2u - \cos 3u + u^3\right]_0^1 = 2 + \sin 2 - \cos 3$$

(b) $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 \left(2u\cos u^2 + 3u^2\sin u^3 - u^4\right)\,du = \left[\sin u^2 - \cos u^3 - \frac{1}{5}\,u^5\right]_0^1 = \frac{4}{5} + \sin 1 - \cos 1$

17. $\mathbf{r}(u) = (1-u)(\mathbf{j} + 4\mathbf{k}) + u(\mathbf{i} - 4\mathbf{k})$

$\quad = u\mathbf{i} + (1-u)\mathbf{j} + (4-8u)\mathbf{k}, \quad u \in [\,0,1\,]$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (-32u + 97u^2 - 64u^3)\,du = \frac{1}{3}$$

19. $\mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j} + u\,\mathbf{k}, \quad u \in [\,0,2\pi\,]$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} \left[-\cos^2 u\,\sin u + \cos^2 u\,\sin u + u^2\right]\,du = \int_0^{2\pi} u^2\,du = \frac{8\pi^3}{3}$$

22. (a) $\mathbf{r}(u) = (1-2u)\mathbf{i}$; $\displaystyle\int_{C_1} \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (1-2u)^2\mathbf{i}\cdot(-2\mathbf{i})\,du = \int_0^1 -2(1-2u)^2\,du = -\frac{2}{3}$

(b) $$\int_{C_2} \mathbf{h} \cdot d\mathbf{r} = \int_0^1 (\mathbf{i} + u\mathbf{j}) \cdot \mathbf{j}\, du + \int_0^1 [(1-2u)^2\mathbf{i} + \mathbf{j}] \cdot (-2\mathbf{i})\, du + \int_0^1 [\mathbf{i} + (1-u)\mathbf{j}] \cdot (-\mathbf{j})\, du$$
$$= \int_0^1 u\, du + \int_0^1 -2(1-2u)^2\, du + \int_0^1 -(1-u)\, du = -\frac{2}{3}$$

(c) $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}, \quad u \in [0, \pi]$

$$\int_{C_3} \mathbf{h} \cdot d\mathbf{r} = \int_0^\pi (\cos^2 u\mathbf{i} + \sin u\mathbf{j}) \cdot (-\sin u\mathbf{i} + \cos u\mathbf{j})\, du = \int_0^\pi (-\sin u \cos^2 u + \sin u \cos u)\, du = -\frac{2}{3}$$

25. $E : \mathbf{r}(u) = a\cos u\, \mathbf{i} + b\sin u\, \mathbf{j}, \quad u \in [\,0, 2\pi\,]$

$$W = \int_0^{2\pi} \left[\left(-\frac{1}{2}b\sin u\right)(-a\sin u) + \left(\frac{1}{2}a\cos u\right)(b\cos u)\right] du = \frac{1}{2}\int_0^{2\pi} ab\, du = \pi ab$$

If the ellipse is traversed in the opposite direction, then $W = -\pi ab$. In both cases $|W| = \pi ab =$ area of the ellipse.

27. $\mathbf{r}(t) = \alpha t\mathbf{i} + \beta t^2\mathbf{j} + \gamma t^3\mathbf{k}$

$\mathbf{r}'(t) = \alpha\mathbf{i} + 2\beta t\mathbf{j} + 3\gamma t^2\mathbf{k}$

force at time $t = m\mathbf{r}''(t) = m(2\beta\mathbf{j} + 6\gamma t\mathbf{k})$

$$W = \int_0^1 [m(2\beta\mathbf{j} + 6\gamma t\mathbf{k}) \cdot (\alpha\mathbf{i} + 2\beta t\mathbf{j} + 3\gamma t^2\mathbf{k})]\, dt$$
$$= m\int_0^1 (4\beta^2 t + 18\gamma^2 t^3)\, dt = \left(2\beta^2 + \frac{9}{2}\gamma^2\right) m$$

SECTION 17.2

3. $\mathbf{h}(x,y) = \nabla f(x,y) \quad \text{where} \quad f(x,y) = x\cos \pi y; \qquad \mathbf{r}(0) = \mathbf{0}, \quad \mathbf{r}(1) = \mathbf{i} - \mathbf{j}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,-1) - f(0,0) = -1$$

4. $\mathbf{h} = \nabla f \quad \text{with} \quad f(x,y) = \dfrac{x^3}{3} + \dfrac{y^3}{3} - xy,$ and C is closed, so $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = 0$

7. $\mathbf{h}(x,y) = \nabla f(x,y) \quad \text{where} \quad f(x,y) = x^2y - xy^2; \quad \mathbf{r}(0) = \mathbf{i}, \ \mathbf{r}(\pi) = -\mathbf{i}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(-1,0) - f(1,0) = 0 - 0 = 0$$

10. $\mathbf{h} = \nabla f \quad \text{with} \quad f(x,y) = \cosh x^2y;$ and C is closed, so $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = 0$

12. $\mathbf{h}(x,y) = \nabla\left(\dfrac{x^2y^2}{2}\right)$

(a) $\displaystyle\int_0^2 (u^5\mathbf{i} + u^4\mathbf{j}) \cdot (\mathbf{i} + 2u\mathbf{j})\, du = \int_0^2 3u^5\, du = 32$

(b) $f(2,4) - f(0,0) = 32 - 0 = 32$

14. $\mathbf{h}(x,y) = \nabla(x^2 \sin y - e^x)$

(a) $\displaystyle\int_0^\pi \left[(2\cos u \sin u - e^{\cos u})\,\mathbf{i} + (\cos^2 u \cos u)\mathbf{j}\right] \cdot (-\sin u\mathbf{i} + \mathbf{j})\, du = e - e^{-1}$

(b) $f(-1,\pi) - f(1,0) = e - e^{-1}$

17. $\mathbf{h}(x,y,z) = (2xz + \sin y)\,\mathbf{i} + x\,\cos y\,\mathbf{j} + x^2\,\mathbf{k};$

$\dfrac{\partial P}{\partial y} = \cos y = \dfrac{\partial Q}{\partial x}, \quad \dfrac{\partial P}{\partial z} = 2x = \dfrac{\partial R}{\partial x}, \quad \dfrac{\partial Q}{\partial z} = 0 = \dfrac{\partial R}{\partial y}.$ Thus $\mathbf{h}$ is a gradient.

$$\frac{\partial f}{\partial x} = 2xz + \sin y, \quad \Longrightarrow \quad f(x,y,z) = x^2z + x\,\sin y + g(y,z)$$

$$\frac{\partial f}{\partial y} = x\,\cos y + \frac{\partial g}{\partial y} = x\,\cos y, \quad \Longrightarrow \quad g(y,z) = h(z) \quad \Longrightarrow f(x,y,z) = x^2z + x\,\sin y + h(z)$$

$$\frac{\partial f}{\partial z} = x^2 + h'(z) = x^2 \quad \Longrightarrow \quad h'(z) = 0 \quad \Longrightarrow \quad h(z) = C$$

Therefore, $f(x,y,z) = x^2z + x\,\sin y$ (take $C = 0$)

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \left[x^2z + x\,\sin y\right]_{\mathbf{r}(0)}^{\mathbf{r}(2\pi)} = \left[x^2z + x\,\sin y\right]_{(1,0,0)}^{(1,0,2\pi)} = 2\pi$$

19. $\mathbf{h}(x,y,z) = (2xy + z^2)\,\mathbf{i} + x^2\,\mathbf{j} + 2xz\,\mathbf{k};$

$\dfrac{\partial P}{\partial y = 2x = }\dfrac{\partial Q}{\partial x,} \quad \dfrac{\partial P}{\partial z = 2z = }\dfrac{\partial R}{\partial x,} \quad \dfrac{\partial Q}{\partial z = 0 = }\dfrac{\partial R}{\partial y.}$ Thus $\mathbf{h}$ is a gradient.

$$\frac{\partial f}{\partial x = 2xy + z^2} \quad \Longrightarrow \quad f(x,y,z) = x^2y + xz^2 + g(y,z)$$

$$\frac{\partial f}{\partial y = x^2} + \frac{\partial g}{\partial y = x^2} \quad \Longrightarrow \quad g(y,z) = h(z) \quad \Longrightarrow f(x,y,z) = x^2y + xz^2 + h(z)$$

$$\frac{\partial f}{\partial z = 2xz + h'(z) = 2xz} \quad \Longrightarrow \quad h'(z) = 0 \quad \Longrightarrow \quad h(z) = C$$

Therefore, $f(x,y,z) = x^2y + xz^2$ (take $C = 0$)

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \left[x^2y + xz^2\right]_{\mathbf{r}(0)}^{\mathbf{r}(1)} = \left[x^2y + xz^2\right]_{(0,2,0)}^{(2,3,-1)} = 14$$

21. $\mathbf{F}(x,y) = (x + e^{2y})\,\mathbf{i} + (2y + 2xe^{2y})\,\mathbf{j}; \quad \dfrac{\partial P}{\partial y} = 2e^{2y} = \dfrac{\partial Q}{\partial x}.$ Thus $\mathbf{F}$ is a gradient.

$$\frac{\partial f}{\partial x} = x + e^{2y} \quad \Longrightarrow \quad f(x,y) = \frac{1}{2}\,x^2 + xe^{2y} + g(y);$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} + g'(y) = 2y + 2xe^{2y} \quad \Longrightarrow \quad g'(y) = 2y \quad \Longrightarrow g(y) = y^2 \quad (\text{take } C = 0)$$

Therefore, $f(x,y) = \dfrac{1}{2}\,x^2 + xe^{2y} + y^2.$

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \left[\frac{1}{2}\,x^2 + xe^{2y} + y^2\right]_{\mathbf{r}(0)}^{\mathbf{r}(2\pi)} = \left[\frac{1}{2}\,x^2 + xe^{2y} + y^2\right]_{(3,0)}^{(3,0)} = 0$$

23. Set $f(x,y,z) = g(x)$ and $C : \mathbf{r}(u) = u\mathbf{i}, \quad u \in [a,b].$

In this case

$$\nabla f(\mathbf{r}(u)) = g'(x(u))\mathbf{i} = g'(u)\mathbf{i} \quad \text{and} \quad \mathbf{r}'(u) = \mathbf{i},$$

so that

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b [\nabla f(\mathbf{r}(u)) \cdot \mathbf{r}'(u)]\,du = \int_a^b g'(u)\,du.$$

Since $\quad f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = g(b) - g(a),$

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad \text{gives} \quad \int_a^b g'(u)\,du = g(b) - g(a).$$

SECTION 17.4

5. $\mathbf{r}(u) = 2u^2\,\mathbf{i} + u\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)\,x'(u) + x(u)\,y(u)\,y'(u)]\,du = \int_0^1 (4u^2 + 2u^3)\,du = \frac{11}{6}$$

6. $\mathbf{r}(u) = 2u\mathbf{i} + u\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)x'(u) + x(u)y(u)y'(u)]\,du$$

$$= \int_0^1 (2u + 2u^2)\,du = \frac{5}{3}$$

7. $C = C_1 \cup C_2 \qquad C_1: \mathbf{r}(u) = u\,\mathbf{j}, \quad u \in [0,1]; \qquad C_2: \mathbf{r}(u) = 2u\,\mathbf{i} + \mathbf{j}, \quad u \in [0,1]$

$$\int_{C_1} y\,dx + xy\,dy = 0$$

$$\int_{C_2} y\,dx + xy\,dy = \int_{C_2} y\,dx = \int_0^1 y(u)\,x'(u)\,du = \int_0^1 2\,du = 2$$

$$\int_C = \int_{C_1} + \int_{C_2} = 2$$

8. $\mathbf{r}(u) = 2u^3\mathbf{i} + u\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)x'(u) + x(u)y(u)y'(u)]\,du$$

$$= \int_0^1 (6u^3 + 2u^4)\,du = \frac{19}{10}$$

13. $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy$$

$$= \int_0^1 \left\{[y^2(u) + 2x(u) + 1]x'(u) + [2x(u)y(u) + 4y(u) - 1]y'(u)\right\}\,du$$

$$\int_0^1 \left[(u^2 + 2u + 1) + (2u^2 + 4u - 1)\right]\,du = \int_0^1 \left(3u^2 + 6u\right)\,du = 4$$

14. $\mathbf{r}(u) = u\mathbf{i} + u^2\mathbf{j}, \quad u \in [0,1].$

$$\begin{aligned}
&\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy \\
&= \int_0^1 \left[(y^2(u) + 2x(u) + 1)x'(u) + (2x(u)y(u) + 4y(u) - 1)y'(u)\right]\,du \\
&= \int_0^1 (5u^4 + 8u^3 + 1)\,du = 4
\end{aligned}$$

15. $\mathbf{r}(u) = u\,\mathbf{i} + u^3\,\mathbf{j}, \quad u \in [\,0,1\,]$

$$\begin{aligned}
&\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy \\
&= \int_0^1 \left\{[y^2(u) + 2x(u) + 1]x'(u) + [2x(u)y(u) + 4y(u) - 1]y'(u)\right\}\,du \\
&= \int_0^1 \left[(u^6 + 2u + 1) + (2u^4 + 4u^3 - 1)3u^2\right]\,du = \int_0^1 \left(7u^6 + 12u^5 - 3u^2 + 2u + 1\right)\,du = 4
\end{aligned}$$

16. $C = C_1 \cup C_2 \cup C_3$

$C_1 : \mathbf{r}(u) = 4u\mathbf{i},\ u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 4\mathbf{i} + 2u\mathbf{j},\ u \in [0,1]; \quad C_3 : \mathbf{r}(u) = (4-3u)\mathbf{i} + (2-u)\mathbf{j},\ u \in [0,1]$

$$\int_{C_1} = \int_{C_1} (y^2 + 2x + 1)\,dx = \int_0^1 4(8u + 1)\,du = 20$$

$$\int_{C_2} = \int_{C_2} (8y + 4y - 1)\,dy = \int_0^1 2(24u - 1)\,du = 22$$

$$\begin{aligned}
\int_{C_3} &= \int_0^1 \left\{-3\left[(2-u)^2 + 2(4-3u) + 1\right] - [2(4-3u)(2-u) + 4(2-u) - 1]\right\}\,du \\
&= \int_0^1 (-9u^2 + 54u - 62)\,du = -38.
\end{aligned}$$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{c_3} = 20 + 22 - 38 = 4.$$

21. $\mathbf{r}(u) = 2u\,\mathbf{i} + 2u\,\mathbf{j} + 8u\,\mathbf{k}, \quad u \in [\,0,1\,]$

$$\begin{aligned}
&\int_C xy\,dx + 2z\,dy + (y + z)\,dz \\
&= \int_0^1 \{x(u)y(u)x'(u) + 2z(u)y'(u) + [y(u) + z(u)]z'(u)\}\,du \\
&= \int_0^1 [(2u)(2u)(2) + 2(8u)(2) + (2u + 8u)(8)]\,du \\
&= \int_0^1 \left(8u^2 + 112u\right)\,du = \frac{176}{3}
\end{aligned}$$

22. $C = C_1 \cup C_2 \cup C_3$

$C_1 : \mathbf{r}(u) = 2u\mathbf{i}, \quad u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i}+2u\mathbf{j}, \quad u \in [0,1]; \quad C_3 : \mathbf{r}(u) = 2\mathbf{i}+2\mathbf{j}+2u\mathbf{k}, \quad u \in [0,1]$

$$\int_{C_1} xy\,dx + 2z\,dy + (y+z)\,dz = 0$$
$$\int_{C_2} xy\,dx + 2z\,dy + (y+z)\,dz = 0$$
$$\int_{C_3} xy\,dx + 2z\,dy + (y+z)\,dz = \int_{C_2} (y+z)\,dz = \int_0^1 2(2+2u)\,du = 6$$

23. $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j} + 2u^2\,\mathbf{k}, \quad u \in [\,0,2\,]$

$$\begin{aligned}
&\int_C xy\,dx + 2z\,dy + (y+z)\,dz \\
&= \int_0^2 \{x(u)y(u)x'(u) + 2z(u)y'(u) + [y(u)+z(u)]z'(u)\}\;du \\
&= \int_0^2 \left[(u)(u)(1) + 2(2u^2)(1) + (u+2u^2)(4u)\right]\;du \\
&= \int_0^2 \left(8u^3 + 9u^2\right)\;du = 56
\end{aligned}$$

24. $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = 2u\mathbf{i} + 2u\mathbf{j} + 2u\mathbf{k},\ u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i} + 2\mathbf{j} + (2+6u)\mathbf{k},\ u \in [0,1].$

$$\int_{C_1} xy\,dx + 2z\,dy + (y+z)\,dz = \int_0^1 8(u^2+2u)\,du = \frac{32}{3}$$
$$\int_{C_2} xy\,dx + 2z\,dy + (y+z)\,dz = \int_{C_2} (y+z)\,dz = \int_0^1 6(4+6u)\,du = 42$$
$$\int_C = \int_{C_1} + \int_{C_2} = \frac{158}{3}$$

27. (a) $\dfrac{\partial P}{\partial y} = 6x - 4y = \dfrac{\partial Q}{\partial x}$

$$\frac{\partial f}{\partial x} = x^2 + 6xy - 2y^2 \quad \Longrightarrow \quad f(x,y) = \frac{1}{3}x^3 + 3x^2y - 2xy^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 3x^2 - 4xy + g'(y) = 3x^2 - 4xy + 2y \quad \Longrightarrow \quad g'(y) = 2y \quad \Longrightarrow g(y) = y^2 + C$$

Therefore, $f(x,y) = \dfrac{1}{3}x^3 + 3x^2y - 2xy^2 + y^2$ (take $C = 0$)

(b) $\displaystyle\int_C (x^2 + 6xy - 2y^2)\,dx + (3x^2 - 4xy + 2y)\,dy = [f(x,y)]_{(3,0)}^{(0,4)} = 7$

(c) $\displaystyle\int_C' (x^2 + 6xy - 2y^2)\,dx + (3x^2 - 4xy + 2y)\,dy = [f(x,y)]_{(4,0)}^{(0,3)} = -\frac{37}{3}$

28. (a) $\mathbf{F} = \nabla f$ where $f(x,y,z) = x^2y + xz^2 - y^2z$

(b) $\int_C (2xy + z^2)\,dx + (x^2 - 2yz)\,dy + (2xz - y^2)\,dz = f(3,2,-1) - f(1,0,1) = 25 - 1 = 24$

(c) $\int_{C'} (2xy + z^2)\,dx + (x^2 - 2yz)\,dy + (2xz - y^2)\,dz = f(1,0,1) - f(3,2-1) = -24$

SECTION 17.5

1. (a) $\oint_C xy\,dx + x^2\,dy = \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy + \int_{C_3} xy\,dx + x^2\,dy,$ where

$C_1:\ \mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j},\quad u \in [0,1];\quad C_2:\ \mathbf{r}(u) = (1-u)\,\mathbf{i} + \mathbf{j},\quad u \in [0,1]$

$C_3:\ \mathbf{r}(u) = (1-u)\,\mathbf{j},\quad u \in [0,1].$

$$\int_{C_1} xy\,dx + x^2\,dy = \int_0^1 (u^2 + u^2)\,du = \frac{2}{3}$$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_0^1 -(1-u)\,du = -\frac{1}{2}$$

$$\int_{C_3} xy\,dx + x^2\,dy = \int_0^1 0^2(-1)\,du = 0$$

Therefore, $\oint_C xy\,dx + x^2\,dy = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$

(b) $\oint_C xy\,dx + x^2\,dy = \iint_\Omega x\,dx\,dy = \int_0^1 \int_0^y x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_0^y du = \frac{1}{2}\int_0^1 y^2\,dy = \frac{1}{6}$

3. (a) $C:\ \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j},\quad u \in [0, 2\pi]$

$$\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy \int_0^{2\pi} \left[(3\cos^2 u + \sin u)(-\sin u) + (2\cos u + \sin^3 u)\cos u\right] du$$

$$= \int_0^{2\pi} \left[3\cos^2 u(-\sin u) - \sin^2 u + 2\cos^2 u + \sin^3 u\ \cos u\right] du$$

$$= \left[\cos^3 u - \frac{1}{2}u + \frac{1}{4}sin 2u + u + \frac{1}{2}\sin 2u + \frac{1}{4}sin^4 u\right]_0^{2\pi} = \pi$$

(b) $\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy = \iint_\Omega 1\,dx\,dy = \text{area } \Omega = \pi$

4. (a) $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = u\mathbf{i} + u^2\mathbf{j},\ u \in [0,1];\quad C_2 : \mathbf{r}(u) = (1-u)\mathbf{i} + (1-u)\mathbf{j},\ u \in [0,1]$

$$\int_{C_1} y^2\,dx + x^2\,dy = \int_0^1 (u^4 + 2u^3)\,du = \frac{7}{10}$$

$$\int_{C_2} y^2\,dx + x^2\,dy = \int_0^1 -2(1-u)^2\,du = -\frac{2}{3}; \quad \oint C = \int_{C_1} + \int_{C_2} = \frac{1}{30}$$

(b) $$\oint Cy^2\,dx + x^2\,dy = \iint_\Omega \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dx\,dy = \int_0^1 \int_{x^2}^{x} 2(x-y)\,dy\,dx = \frac{1}{30}$$

5. $$\oint_C 3y\,dx + 5x\,dy = \iint_\Omega (5-3)\,dxdy = 2A = 2\pi$$

7. $$\oint_C x^2\,dy = \iint_\Omega 2x\,dxdy = 2\overline{x}A = 2\left(\frac{a}{2}\right)(ab) = a^2 b$$

9. $$\oint_C (3xy + y^2)\,dx + (2xy + 5x^2)\,dy = \iint_\Omega [(2y + 10x) - (3x + 2y)]\,dxdy$$
$$= \iint_\Omega 7x\,dxdy = 7\,\overline{x}A = 7(1)(\pi) = 7\pi$$

13. $$\oint_C e^x \sin y\,dx + e^x \cos y\,dy = \iint_\Omega [e^x \cos y - e^x \cos y]\,dxdy = 0$$

15. $$\oint_C 2xy\,dx + x^2\,dy = \iint_\Omega [2x - 2x]\,dxdy = 0$$

17. $C:\ \mathbf{r}(u) = a\,\cos u\,\mathbf{i} + a\,\sin u\,\mathbf{j};\quad u \in [\,0, 2\pi\,]$

$$A = \oint_C -y\,dx = \int_0^{2\pi} (-a\,\sin u)(-\,\sin u)\,du = a^2 \int_0^{2\pi} \sin^2 u\,du = a^2 \left[\frac{1}{2}\,u - \frac{1}{4}\,\sin 2u\right]_0^{2\pi} = \pi\,a^2$$

19. $A = \oint_C x\,dy,$ where $C = C_1 \cup C_2;$

$C_1:\mathbf{r}(u) = u\,\mathbf{i} + \dfrac{4}{u}\,\mathbf{j},\ 1 \le u \le 4; \quad C_2:\mathbf{r}(u) = (4-3u)\,\mathbf{i} + (1+3u)\,\mathbf{j},\ 0 \le u \le 1.$

$$\oint_{C_1} x\,dy = \int_1^4 u\left(\frac{-4}{u^2}\right)du = -4\int_1^4 \frac{1}{u}\,du = -4\ln 4;$$

$$\oint_{C_2} x\,dy = \int_0^1 (4-3u)3\,du = \int_0^1 (12-9u)\,du = \frac{15}{2}.$$

Therefore, $\quad A = \frac{15}{2} - 4\ln 4.$

20. $A = \dfrac{1}{2}\oint_C x\,dy - y\,dx,$ where $C = C_1 \cup C_2;$

$C_1:\mathbf{r}(u) = \sqrt{5}\,\tan u\,\mathbf{i} + \sqrt{5}\,\sec u\,\mathbf{j},\ \tan^{-1}\left(-2/\sqrt{5}\right) \le u \le \tan^{-1}\left(2/\sqrt{5}\right)$

$C_2 : (2-4u)\,\mathbf{i} + 3\,\mathbf{j},\ \ 0 \le u \le 1$

$\frac{1}{2}\oint_{C_1} x\,dy - y\,dx = \frac{5}{2}\ln 5, \quad \frac{1}{2}\oint_{C_2} x\,dy - y\,dx = 6 \quad$ Therefore, $\quad A = 6 + \frac{5}{2}\ln 5.$

31. $\dfrac{\partial P}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2} = \dfrac{\partial Q}{\partial x}$ except at $(0,0)$

(a) If C does not enclose the origin, and Ω is the region enclosed by C, then

$$\oint_C \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy = \iint_\Omega 0\,dxdy = 0.$$

(b) If C does enclose the origin, then

$$\oint_C = \oint_{C_a}$$

where $C_a : \mathbf{r}(u) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$ is a small circle in the inner region of C.

In this case

$$\oint_C = \int_0^{2\pi}\left[\frac{a\cos u}{a^2}(-a\sin u) + \frac{a\sin u}{a^2}(a\cos u)\right] du = \int_0^{2\pi} 0\,du = 0.$$

The integral is still 0.

32. (a) $\oint C - \dfrac{y^3}{(x^2+y^2)^2}\,dx + \dfrac{xy^2}{(x^2+y^2)^2}\,dy = \iint_\Omega 0\,dy\,dx = 0$

(b) By Green's theorem, $\oint C = \oint_{C'}$, where C' is a circle about the origin. $\mathbf{r}(u) = a\cos u\mathbf{i} + a\sin u\mathbf{j}$.

$\oint_{C'} = \int_0^{2\pi}(\sin^4 u + \sin^2 u\cos^2 u)\,du = \int_0^{2\pi}\sin^2 u\,du = \pi$