

3-D Coordinate System: (12.1)

$P : (x_1, y_1, z_1)$ and $Q : (x_2, y_2, z_2)$, points in 3-space:

(A) Distance Formula:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

(B) Midpoint Formula: The midpoint R of the line segment joining P and Q is the point

$$R : \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

(C) Equation for the sphere of radius r and center $P(a, b, c)$:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Examples:

1. The points

$$A : (2, -1, 3) \quad \text{and} \quad B : (-4, 5, -1)$$

are the endpoints of a diameter of a sphere.
Find an equation for the sphere.

2. The points

$$A : (2, -2, 1), \quad B : (1, 1, 3), \quad C : (2, 0, 5)$$

are the vertices of a right triangle. Find an equation of the sphere with center at the midpoint of the hypotenuse and passing through the vertex opposite the hypotenuse.

Vectors: (12.2, 12.3)

A *vector* in n -dimensional space is a directed line segment; it is represented by an ordered n -tuple of real numbers.

3-Space: A vector \mathbf{a} in 3-space is an ordered triple of numbers:

$$\mathbf{a} = (a_1, a_2, a_3)$$

The vector $\mathbf{0} = (0, 0, 0)$ is the *zero vector*.

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in 3-space and let α be a real number (scalar).

(A) Equality: $\mathbf{a} = \mathbf{b}$ iff

$$a_1 = b_1, a_2 = b_2, a_3 = b_3.$$

(B) Vector Addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

Motivation from physics:

Properties of Addition: Let a, b, c be vectors.

1. $a + b = b + a$ (commutative)

2. $(a + b) + c = a + (b + c)$ (associative)

3. $a + 0 = 0 + a = a$, 0 is the additive identity

4. to each vector a there corresponds a unique x such that

$$a + x = x + a = 0 \quad (\text{additive inverse})$$

x is denoted $-a$

Subtraction:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

(C) Multiplication by a Scalar:

$$\alpha \mathbf{a} = (\alpha a_1, \alpha a_2, \alpha a_3).$$

Motivation from Physics:

Properties of Mult. by a Scalar:

1. $1 \mathbf{a} = \mathbf{a}, \quad 0 \mathbf{a} = \mathbf{0}$

2. $(\alpha + \beta)\mathbf{a} = \alpha \mathbf{a} + \beta \mathbf{a},$

$\alpha(\mathbf{a} + \mathbf{b}) = \alpha \mathbf{a} + \alpha \mathbf{b}$ distributive laws

3. $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a}) = \beta(\alpha\mathbf{a})$

NOTE: \mathbf{a} and \mathbf{b} are *parallel* iff $\mathbf{a} = \lambda\mathbf{b}$
for some number λ .

$\mathbf{0}$ is parallel to every vector; $\mathbf{0} = 0 \mathbf{a}$ for
all \mathbf{a} .

(D) Norm (Magnitude) & Direction:

The *norm* of \mathbf{a} , denoted by $\|\mathbf{a}\|$, is:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad \|\mathbf{0}\| = 0.$$

$\|\mathbf{a}\|$ is a nonnegative number; it is the length of the vector \mathbf{a} .

Properties of Norm:

1. $\|\mathbf{a}\| \geq 0$; $\|\mathbf{a}\| = 0$ iff $\mathbf{a} = \mathbf{0}$.
2. $\|\alpha\mathbf{a}\| = |\alpha| \|\mathbf{a}\|$.
3. $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. (triangle inequality)

Unit Vectors:

\mathbf{u} is a *unit vector* if $\|\mathbf{u}\| = 1$

If \mathbf{b} is a non-zero vector, then

$$\mathbf{u}_b = \frac{1}{\|\mathbf{b}\|} \mathbf{b}$$

is a unit vector in the same direction as \mathbf{b} .

Unit Coordinate Vectors:

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ -Representation:

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

Direction: ????

$\mathbf{0}$ has no direction.

Dot Product: (12.4)

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors. The *dot product* of \mathbf{a} and \mathbf{b} , denoted $\mathbf{a} \cdot \mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Properties of Dot Product:

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$
2. $\mathbf{0} \cdot \mathbf{a} = 0.$
3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$ (commutative)
4. $(\alpha \mathbf{a}) \cdot (\beta \mathbf{b}) = \alpha\beta(\mathbf{a} \cdot \mathbf{b}).$
5. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive)

Applications of the Dot Product:

1. Angle between two vectors:

$$0 \leq \theta \leq \pi$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta, \quad 0 \leq \theta \leq \pi.$$

or

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \mathbf{u}_a \cdot \mathbf{u}_b.$$

Special cases:

a. \mathbf{a} and \mathbf{b} have the same direction iff $\theta = 0$, and

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\|.$$

b. \mathbf{a} and \mathbf{b} are perpendicular iff $\theta = \pi/2$, and

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

NOTE: $\mathbf{0} \cdot \mathbf{a} = 0$, i.e., $\mathbf{0}$ is perpendicular to every vector.

c. \mathbf{a} and \mathbf{b} have opposite direction iff $\theta = \pi$, and

$$\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\| \|\mathbf{b}\|.$$

Components and Projections

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \|\mathbf{a}\| \cos \theta = \frac{\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta}{\|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \mathbf{u}_{\mathbf{b}} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \right) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}$$

Special Case: \mathbf{b} is a unit vector:

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \mathbf{a} \cdot \mathbf{b}, \quad \text{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{b}$$

Direction Angles:

Given $\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$.

Let α be the angle between \mathbf{a} and \mathbf{i} , β the angle between \mathbf{a} and \mathbf{j} , and γ the angle between \mathbf{a} and \mathbf{k} .

α, β, γ are the *direction angles* of \mathbf{a} .

Direction Cosines:

$$\cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{a_3}{\|\mathbf{a}\|}$$

Note: $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

c.f. $\sin^2 \theta + \cos^2 \theta = 1$

Cross Product: (12.5)

NOTE: The cross product is defined only in 3-space.

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors, $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a} \neq \lambda \mathbf{b}$.

The *cross product* of \mathbf{a} and \mathbf{b} , denoted $\mathbf{a} \times \mathbf{b}$ is the vector satisfying

1. $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by \mathbf{a} and \mathbf{b} .
2. \mathbf{a} , \mathbf{b} , $\mathbf{a} \times \mathbf{b}$, in that order, form a right-handed system.
3. $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$

Properties of the Cross Product:

1. If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, or $\mathbf{a} = \lambda \mathbf{b}$, then,

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

2. $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ (anti-commutative)

3. Not associative!!

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$$

4. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

5. $\alpha(\mathbf{a} \times \mathbf{b}) = (\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b})$

The Components of $\mathbf{a} \times \mathbf{b}$:

Special case: $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$(\mathbf{a}_1 \mathbf{i} + \mathbf{a}_2 \mathbf{j} + \mathbf{a}_3 \mathbf{k}) \times (\mathbf{b}_1 \mathbf{i} + \mathbf{b}_2 \mathbf{j} + \mathbf{b}_3 \mathbf{k}) =$$

$$(\mathbf{a}_2 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_2) \mathbf{i} - (\mathbf{a}_1 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_1) \mathbf{j} + (\mathbf{a}_1 \mathbf{b}_2 - \mathbf{a}_2 \mathbf{b}_1) \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{vmatrix}.$$

Examples

1. $\mathbf{a} = (2, 3, -1)$, $\mathbf{b} = (4, 0, 5)$. Find $\mathbf{a} \times \mathbf{b}$.

2. Given points

$$P : (1, -1, 4), \quad Q : (2, 0, 1), \quad R : (0, 2, 3).$$

a. Show that the points are not collinear

b. Find a vector which is perpendicular to the plane determined by the points.

Triple Scalar Product:

- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the *triple scalar product*
- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$
- \mathbf{a} and \mathbf{b} determine a plane \mathcal{P} . The vectors \mathbf{a} , \mathbf{b} , \mathbf{c} form a *right-handed system* if \mathbf{c} is on the same side of \mathcal{P} as $\mathbf{a} \times \mathbf{b}$.
- If \mathbf{a} , \mathbf{b} , \mathbf{c} is a right-handed system, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} > 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \text{volume of "box" with sides } \mathbf{a}, \mathbf{b}, \mathbf{c}$
- If \mathbf{a} , \mathbf{b} , \mathbf{c} is not a right-handed system, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} < 0$

$$\bullet (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Examples:

1. Find the volume of the “box” having

$$\mathbf{a} = (1, -3, 1), \mathbf{b} = (0, 2, -1), \mathbf{c} = (1, 1, -2)$$

as sides.

2. Which of the following makes sense?

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}), \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}), \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

Lines: (12.6)

Given points $P : (x_0, y_0, z_0)$ and $Q : (x_1, y_1, z_1)$, $P \neq Q$.

There is one and only one straight line \mathcal{L} passing through P and Q .

Direction vector:

$$\mathbf{d} = (x_1 - x_0, y_1 - y_0, z_1 - z_0) = (d_1, d_2, d_3)$$

Equations:

1. Vector equation: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}$.

2. Scalar parametric equations:

$$x = x_0 + d_1t, \quad y = y_0 + d_2t, \quad z = z_0 + d_3t.$$

3. Symmetric equations:

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}.$$

Examples:

\mathcal{L} The line determined by

$$P : (-1, 3, 2), \quad Q : (2, 4, -2).$$

1. Direction vector for \mathcal{L} :
2. Direction numbers for \mathcal{L} :
3. Vector equation for \mathcal{L} :
4. Parametric equations for \mathcal{L} :
5. Symmetric equations for \mathcal{L} :

Two lines; Intersection:

Given two lines

$$\mathcal{L}_1 : \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}, \quad \mathcal{L}_2 : \mathbf{R}(s) = \mathbf{R}_0 + s\mathbf{D}.$$

1. Parallel or coincident if $\mathbf{d} = \lambda\mathbf{D}$.

Examples

a.

$$\mathcal{L}_1 : x = 2 - 2t, \quad y = 3 + t, \quad z = -1 + 3t,$$

$$\mathcal{L}_2 : x = 4 + 4s, \quad y = -2s, \quad z = 6 - 6s$$

b.

$$\mathcal{L}_1 : x = 2 - 2t, \quad y = 3 + t, \quad z = -1 + 3t,$$

$$\mathcal{L}_2 : x = -2 + 4s, \quad y = 5 - 2s, \quad z = 5 - 6s$$

2. Intersecting? $d \neq \lambda D$.

a.

$$\mathcal{L}_1 : x = -1 + 3t, y = 3 + t, z = 2 - 4t,$$

$$\mathcal{L}_2 : x = 1 - s, y = 7 + 3s, z = 2s.$$

b.

$$\mathcal{L}_1 : x = -1 + 3t, y = 3 + t, z = 2 - 4t,$$

$$\mathcal{L}_2 : x = 4 - s, y = 2 + 2s, z = -1 + s.$$

Lines that do not intersect and are not parallel are called **skew lines**.

Angle between two lines:

$$0 \leq \alpha \leq \pi/2.$$

$$\cos \alpha = \frac{|\mathbf{d} \cdot \mathbf{D}|}{\|\mathbf{d}\| \|\mathbf{D}\|}.$$

Example:

$$\mathcal{L}_1 : x = -1 + 3t, \quad y = 3 + t, \quad z = 2 - 4t,$$

$$\mathcal{L}_2 : x = 1 - s, \quad y = 7 + 3s, \quad z = 2s$$

$$\cos \alpha = \frac{|(3, 1, -4) \cdot (-1, 3, 2)|}{\sqrt{26} \sqrt{14}} = \frac{8}{2\sqrt{91}}$$

**Distance from a point $P : (x_1, y_1, z_1)$ to
a line \mathcal{L} :**

$$d(P, \mathcal{L}) = \frac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}.$$

Example:

$$P : (1, 2, 3),$$

$$\mathcal{L} : x = 1 + t, \quad y = -2t, \quad z = 2 + 3t$$

Planes: (12.7)

Given a point P and a vector \mathbf{N} . There is one and only one plane \mathcal{P} which is perpendicular to \mathbf{N} and passes through P .

Equations:

Given $P : (x_0, y_0, z_0)$ and

$$\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}.$$

The set of all points $Q : (x, y, z)$ such that

$$\mathbf{N} \cdot \overrightarrow{PQ} = 0$$

is a plane \mathcal{P} . \mathbf{N} is a **normal** (vector) to \mathcal{P} .

1. Vector equation: $\mathbf{N} \cdot \overrightarrow{PQ} = 0$.

2. Standard form equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

c.f. Point-slope equation for a straight line.

3. Remove parentheses:

$$Ax + By + Cz + D = 0$$

or

$$Ax + By + Cz = E.$$

To find an equation for a plane, you need:

1. A point P which lies in the plane.

2. A normal vector \mathbf{N} .

Examples:

1. Find an equation for the plane which is parallel to the plane

$$5x - 2y + 3z = 4$$

and passes through $P : (2, 3, -1)$.

2. Find an equation for the plane which is perpendicular to the line

$$\mathcal{L} : x = 1 + 6t, y = 2 - 2t, z = 4 + 5t$$

and passes through $P : (-7, 4, 2)$.

3. a. Show that the points

$$P : (3, -4, 1), \quad Q : (4, 2, -1), \quad R : (-1, 1, -2)$$

are not collinear.

b. Find an equation for the plane determined by P , Q , R .

Intersections:

Given two planes \mathcal{P}_1 and \mathcal{P}_2 . Exactly one of the following holds:

1. $\mathcal{P}_1, \mathcal{P}_2$ are parallel. ($\mathbf{N}_1 = \lambda\mathbf{N}_2$)
2. $\mathcal{P}_1, \mathcal{P}_2$ coincide. ($\mathbf{N}_1 = \lambda\mathbf{N}_2$)
3. $\mathcal{P}_1, \mathcal{P}_2$ intersect in a straight line \mathcal{L} .

$$\mathbf{N}_1 \neq \lambda\mathbf{N}_2$$

$\mathbf{d} = \mathbf{N}_1 \times \mathbf{N}_2$ is a direction vector for \mathcal{L} .

Examples:

1. Determine whether the planes

$$\mathcal{P}_1 : -2x + 4y + 6z = 5, \quad \mathcal{P}_2 : x - 2y - 3z - 3 = 0$$

are parallel, coincident, or intersect in a straight line

2. Determine whether the planes

$$\mathcal{P}_1 : (x + 1) - 2(y - 1) - 3(z - 2) = 0,$$

$$\mathcal{P}_2 : -2x + 4y + 6z = 18$$

are parallel, coincident, or intersect in a straight line

3. Prove that the planes

$$\mathcal{P}_1 : 4x + 4y - 2z = 9, \quad \mathcal{P}_2 : 2x + y + z = -3$$

intersect in a straight line and find parametric equations for the line of intersection.

Angle between two planes:

$$\mathcal{P}_1 : A_1x + B_1y + C_1z = D_1,$$

$$\mathcal{P}_2 : A_2x + B_2y + C_2z = D_2$$

$$\mathbf{N}_1 = A_1 \mathbf{i} + B_1 \mathbf{j} + C_1 \mathbf{k}$$

$$\mathbf{N}_2 = A_2 \mathbf{i} + B_2 \mathbf{j} + C_2 \mathbf{k}$$

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|}$$

Example:

Find the cosine of the angle between

$$\mathcal{P}_1 : x + 2y + 3z = 2, \quad \mathcal{P}_2 : -3x + 4y + z = 4$$

Distance from a point to a plane:

Given a plane

$$\mathcal{P} : Ax + By + Cz + D = 0$$

and a point $P : (x_1, y_1, z_1)$ which does not lie on \mathcal{P} .

$$\begin{aligned} d(P_1, \mathcal{P}) &= | \|\overrightarrow{P_0P_1}\| \cos \theta | = \frac{|\|\overrightarrow{P_0P_1}\| \|\mathbf{N}\| \cos \theta|}{\|\mathbf{N}\|} \\ &= \frac{|\overrightarrow{P_0P_1} \cdot \mathbf{N}|}{\|\mathbf{N}\|} \end{aligned}$$

$$d(P_1, \mathcal{P}) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Example:

Show that the point $P : (4, 1 - 3)$ does not lie on the plane

$$\mathcal{P} : 2x + 3y - 4z + 27 = 0$$

and find the distance from P to \mathcal{P} .