

# COORDINATE SYSTEMS

A basic relation between abstract objects (numbers, ordered  $n$ -tuples of numbers) and geometric objects (line, plane, 3-dimensional space, etc.)

## 1-D Coordinate System:

### The Real Line, $\mathbb{R}^1$

points, distance, mid-point, intervals

## 2-D Coordinate System:

### The Plane, $\mathbb{R}^2$

points, distance, mid-point, circles

## 3-D Coordinate System: (12.1)

3-Space, or Space,  $\mathbb{R}^3$

**Points:** Ordered triples of real numbers.

$P : (x_1, y_1, z_1)$  and  $Q : (x_2, y_2, z_2)$ ,

points in 3-space:

(A) **Distance Formula:**

$d(P, Q) =$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(B) **Midpoint Formula:**

The midpoint  $R$  of the line segment joining  $P$  and  $Q$  is the point

$$R : \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

(C) **Equation for a sphere:**

Given a point  $P(a, b, c)$  and a positive number  $r$ . The set of all points  $Q : (x, y, z)$  such that the distance from  $Q$  to  $P$  is  $r$  is a **sphere**  $S$ .

The **standard form** equation for  $S$  is:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

This equation can also be written

$$x^2 + y^2 + z^2 + Ax + By + Cz = D$$

## Examples:

1.  $(x - 3)^2 + (y + 4)^2 + (z - 1)^2 = 5$

is an equation for a sphere.

Give the center and the radius.

2. Given the equations

$$(a) \quad x^2 + y^2 + z^2 - 4x + 6y + 2z - 2 = 0$$

$$(b) \quad x^2 + y^2 + z^2 + 8x - 4y - 2z + 21 = 0$$

$$(c) \quad x^2 + y^2 + z^2 - 6x + 2y - 8z + 20 = 0$$

Each of these is an equation for a sphere??

If so, give the center and radius.

**3.** The points

$A : (2, -1, 3)$  and  $B : (-4, 5, -1)$

are the endpoints of a diameter of a sphere. Find an equation for the sphere.

4. The points

$A : (2, -2, 1)$ ,  $B : (1, 1, 3)$ ,  $C : (2, 0, 5)$

are the vertices of a right triangle.

Find an equation of the sphere with center at the midpoint of the hypotenuse and passing through the vertex opposite the hypotenuse.

## Vectors: (12.2, 12.3)

(Physics) A **vector** is a quantity that has magnitude and direction.

(Mathematics) A **vector** in  $n$ -dimensional space is a directed line segment; it is represented by an ordered  $n$ -tuple of real numbers.

In particular:

**2-Space:** A vector  $\mathbf{a}$  in the plane is an ordered pair of numbers:

$$\mathbf{a} = (a_1, a_2)$$

The vector  $\mathbf{0} = (0, 0)$  is the **zero vector**.

**3-Space:** A vector  $\mathbf{a}$  in 3-space is an ordered triple of numbers:

$$\mathbf{a} = (a_1, a_2, a_3)$$

The vector  $\mathbf{0} = (0, 0, 0)$  is the **zero vector**.

## Operations on Vectors;

### Arithmetic of Vectors

Let  $\mathbf{a} = (a_1, a_2, a_3)$  and  $\mathbf{b} = (b_1, b_2, b_3)$  be vectors in 3-space and let  $\alpha$  be a real number (scalar).

**(A) Equality:**  $\mathbf{a} = \mathbf{b}$  iff

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3.$$

## **(B) Vector Addition:**

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

**Motivation from physics:**

**Properties of Addition:** Let  $a$ ,  $b$ ,  $c$  be vectors.

1.  $a + b = b + a$  (commutative)

2.  $(a + b) + c = a + (b + c)$  (associative)

3.  $a + 0 = 0 + a = a$ ,  $0$  is the additive identity

4. to each vector  $a$  there corresponds a unique  $x$  such that

$$a+x = x+a = 0 \quad (\text{additive inverse})$$

$x$  is denoted  $-a$

## **Subtraction:**

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

## **“Geometric” Representation:**

## **(C) Multiplication by a Scalar:**

$$\alpha \mathbf{a} = (\alpha a_1, \alpha a_2, \alpha a_3).$$

## **Motivation from Physics:**

## Properties of Mult. by a Scalar:

1.  $1 \mathbf{a} = \mathbf{a}$

2.  $0 \mathbf{a} = \mathbf{0}$

### 3. Distributive Laws

$$(\alpha + \beta)\mathbf{a} = \alpha \mathbf{a} + \beta \mathbf{a},$$

$$\alpha(\mathbf{a} + \mathbf{b}) = \alpha \mathbf{a} + \alpha \mathbf{b}$$

4.  $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a}) = \beta(\alpha\mathbf{a})$

## NOTES:

1.  $\mathbf{a}$  and  $\mathbf{b}$  are **parallel** iff

$\mathbf{a} = \lambda \mathbf{b}$  for some number  $\lambda$ .

2.  $\mathbf{0}$  is parallel to every vector;

$\mathbf{0} = 0 \mathbf{a}$  for all  $\mathbf{a}$ .

$\mathbf{0}$  has no direction!!

## (D) Norm (Magnitude):

The **norm** of  $\mathbf{a}$ , denoted by  $\|\mathbf{a}\|$ ,

is:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad \|\mathbf{0}\| = 0.$$

$\|\mathbf{a}\|$  is a nonnegative number; it is the **length** of the vector  $\mathbf{a}$ .

## Properties of Norm:

1.  $\|\mathbf{a}\| \geq 0$ ;  $\|\mathbf{a}\| = 0$  iff  $\mathbf{a} = \mathbf{0}$ .

2.  $\|\alpha\mathbf{a}\| = |\alpha| \|\mathbf{a}\|$ .

3.  $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ . (triangle inequality)

## Unit Vectors:

$\mathbf{u}$  is a **unit vector** if  $\|\mathbf{u}\| = 1$

If  $\mathbf{b}$  is a non-zero vector, then

$$\mathbf{u}_b = \frac{1}{\|\mathbf{b}\|} \mathbf{b}$$

is a unit vector in the same direction

as  $\mathbf{b}$ .

## Unit Coordinate Vectors:

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

## $\mathbf{i}, \mathbf{j}, \mathbf{k}$ -Representation:

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

## Examples:

1. Find  $\alpha$  such that

$$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{b} = \alpha\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

are parallel

2. Find  $\alpha$  such that

$$\mathbf{a} = \alpha\mathbf{i} + (\alpha + 1)\mathbf{j} - (\alpha - 1)\mathbf{k}$$

has length  $\sqrt{29}$ .

3. Find a vector of norm 2 in the direction opposite to

$$\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$$

## (E) **Dot Product:** (12.4)

Let  $\mathbf{a} = (a_1, a_2, a_3)$  and  $\mathbf{b} = (b_1, b_2, b_3)$

be vectors.

The **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$ ,

denoted  $\mathbf{a} \cdot \mathbf{b}$  is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

**Synonyms:** inner product, scalar

product

## Properties of Dot Product:

1.  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$

2.  $\mathbf{0} \cdot \mathbf{a} = 0.$

3.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$  (commutative)

4.  $(\alpha \mathbf{a}) \cdot (\beta \mathbf{b}) = \alpha\beta(\mathbf{a} \cdot \mathbf{b}).$

5.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$  (distributive)

# Applications of the Dot Product:

## 1. Angle between two vectors:

$$0 \leq \theta \leq \pi$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta, \quad 0 \leq \theta \leq \pi.$$

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \mathbf{u}_a \cdot \mathbf{u}_b.$$

$$\theta = \cos^{-1} \left[ \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right]$$

## Special cases:

**a.**  $\mathbf{a}$  and  $\mathbf{b}$  have the same direction iff  $\theta = 0$ , and

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\|.$$

**b.**  $\mathbf{a}$  and  $\mathbf{b}$  have opposite direction iff  $\theta = \pi$ , and

$$\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\| \|\mathbf{b}\|.$$

c.  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular iff

$\theta = \pi/2$ , and

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

**NOTE:**  $\mathbf{0} \cdot \mathbf{a} = 0$ , i.e.,  $\mathbf{0}$  is perpendicular to every vector.

$\mathbf{0}$  has no direction!!

## 2. Components and Projections

$$\begin{aligned}\text{comp}_{\mathbf{b}} \mathbf{a} &= \|\mathbf{a}\| \cos \theta = \frac{\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta}{\|\mathbf{b}\|} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{b}} \mathbf{a} &= (\text{comp}_{\mathbf{b}} \mathbf{a}) \mathbf{u}_{\mathbf{b}} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \right) \frac{\mathbf{b}}{\|\mathbf{b}\|} \\ &= \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}\end{aligned}$$

**Special Case:**  $\mathbf{b}$  is a unit vector:

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \mathbf{a} \cdot \mathbf{b}, \quad \text{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{b}$$

### 3. Direction Angles & Direction

#### Cosines:

Given

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

Let:

$\alpha$  be the angle between  $\mathbf{a}$  and  $\mathbf{i}$ ,

$\beta$  the angle between  $\mathbf{a}$  and  $\mathbf{j}$ , and

$\gamma$  the angle between  $\mathbf{a}$  and  $\mathbf{k}$ .

$\alpha, \beta, \gamma$  are the **direction angles** of **a**.

$$\cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{a_3}{\|\mathbf{a}\|}$$

are the **direction cosines** of **a**.

**Note:**  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$

c.f.  $\sin^2 \theta + \cos^2 \theta = 1$

## Examples:

1. Find the angle between the diagonal of a cube and an edge.

2. Find the angle between the diagonal of a cube and the diagonal of a face.

3. Find a unit vector  $\mathbf{u}$  which is perpendicular to

$$\mathbf{a} = (1, 2, 1) \quad \text{and} \quad \mathbf{b} = (3, -4, 2)$$

4. Does there exist a vector  $\mathbf{a}$  with direction angles

$$3\pi/4, 2\pi/3, \pi/6 \quad ?$$

5. Does there exist a vector  $\mathbf{a}$  of length 2 with direction angles

$$5\pi/4, \pi/3, 5\pi/3 \quad ?$$

If so, what is  $\mathbf{a}$ ?

(F) **Cross Product:** (12.5)

**NOTE: The cross product is defined only in 3-space.**

Let  $\mathbf{a} = (a_1, a_2, a_3)$  and  $\mathbf{b} = (b_1, b_2, b_3)$

be vectors,  $\mathbf{a} \neq \mathbf{0}$ ,  $\mathbf{b} \neq \mathbf{0}$  and

$\mathbf{a} \neq \lambda \mathbf{b}$ .

The **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$ ,

denoted  $\mathbf{a} \times \mathbf{b}$  is the vector satis-

fying:

**1.**  $\mathbf{a} \times \mathbf{b}$  is perpendicular to the plane determined by  $\mathbf{a}$  and  $\mathbf{b}$ .

**2.**  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{a} \times \mathbf{b}$ , in that order, form a right-handed system.

**3.**  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$

= area of parallelogram with sides  $\mathbf{a}$ ,  $\mathbf{b}$ .

## Properties of the Cross Product:

1. If  $\mathbf{a} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$ , or  $\mathbf{a} = \lambda \mathbf{b}$ ,

then,

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

2.  $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$  (anti-commutative)

3. Not associative!!

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$$

$$4. \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

$$5. \quad \alpha(\mathbf{a} \times \mathbf{b}) = (\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b})$$

## The Components of $\mathbf{a} \times \mathbf{b}$ :

Special cases:  $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

## General case:

$$(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) =$$

$$(a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

## Examples

**1.**  $\mathbf{a} = (2, 3, -1), \quad \mathbf{b} = (4, 0, 5).$

Find  $\mathbf{a} \times \mathbf{b}.$

**2.** Given points

$P : (1, -1, 4), \quad Q : (2, 0, 1), \quad R : (0, 2, 3).$

**a.** Show that the points are not collinear

**b.** Find a vector perpendicular to the plane determined by the points.

## Triple Scalar Product:

- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$  is called the **triple scalar product**

- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

- $\mathbf{a}$  and  $\mathbf{b}$  determine a plane  $\mathcal{P}$ .

The vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  form a *right-handed system* if  $\mathbf{c}$  is on the same side of  $\mathcal{P}$  as  $\mathbf{a} \times \mathbf{b}$ .

- If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is a right-handed system, then  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} > 0$  and

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} =$  volume of “box” with sides  $\mathbf{a}$ ,

- If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is not a right-handed system, then  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} < 0$

- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

## Examples:

1. Find the volume of the “box”  
having

$$\mathbf{a} = (1, -3, 1), \mathbf{b} = (0, 2, -1), \mathbf{c} = (1, 1, -2)$$

as sides.

2. Which of the following makes  
sense?

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}), \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}), \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

## Lines: (12.6)

Given points  $P : (x_0, y_0, z_0)$  and  $Q : (x_1, y_1, z_1)$ ,  $P \neq Q$ . There is one and only one straight line  $\mathcal{L}$  passing through  $P$  and  $Q$ .

## Direction vector for $\mathcal{L}$ :

$$\mathbf{d} = (x_1 - x_0, y_1 - y_0, z_1 - z_0) = (d_1, d_2, d_3)$$

## Equations for $\mathcal{L}$ :

1. **Vector equation:**  $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}$ .

2. **Scalar parametric equations:**

$$x = x_0 + d_1t, \quad y = y_0 + d_2t, \quad z = z_0 + d_3t.$$

3. **Symmetric equations:**

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}.$$

**Examples:**  $\mathcal{L}$  The line determined

by  $P : (-1, 3, 2)$ ,  $Q : (2, 4, -2)$ .

1. Direction vector for  $\mathcal{L}$ :

2. Direction numbers for  $\mathcal{L}$ :

3. Vector equation for  $\mathcal{L}$ :

4. Parametric equations for  $\mathcal{L}$ :

5. Symmetric equations for  $\mathcal{L}$ :

## Pairs of Lines; Intersection:

Given two lines

$$\mathcal{L}_1 : \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}, \quad \mathcal{L}_2 : \mathbf{R}(s) = \mathbf{R}_0 + s\mathbf{D}.$$

**A. Parallel or coincident** if

$$\mathbf{d} = \lambda \mathbf{D}.$$

## Examples:

1.

$$\mathcal{L}_1 : x = 2 - 2t, y = 3 + t, z = -1 + 3t,$$

$$\mathcal{L}_2 : x = 4 + 4s, y = -2s, z = 6 - 6s$$

2.

$$\mathcal{L}_1 : x = 2 - 2t, y = 3 + t, z = -1 + 3t,$$

$$\mathcal{L}_2 : x = -2 + 4s, y = 5 - 2s, z = 5 - 6s$$

## B. Intersecting? $d \neq \lambda D$ .

### Examples:

1.

$\mathcal{L}_1$ :

$$x = -1 + 3t, \quad y = 3 + t, \quad z = 2 - 4t,$$

$\mathcal{L}_2$ :

$$x = 1 - s, \quad y = 7 + 3s, \quad z = 2s.$$

2.

$\mathcal{L}_1$ :

$$x = -1 + 3t, \quad y = 3 + t, \quad z = 2 - 4t,$$

$\mathcal{L}_2$ :

$$x = 4 - s, \quad y = 2 + 2s, \quad z = -1 + s.$$

Lines that do not intersect and are not parallel are called **skew lines**.

## Angle between two lines:

$$0 \leq \alpha \leq \pi/2.$$

$$\cos \alpha = \frac{|\mathbf{d} \cdot \mathbf{D}|}{\|\mathbf{d}\| \|\mathbf{D}\|}.$$

## Example:

$\mathcal{L}_1$ :

$$x = -1 + 3t, \quad y = 3 + t, \quad z = 2 - 4t,$$

$\mathcal{L}_2$ :

$$x = 1 - s, \quad y = 7 + 3s, \quad z = 2s.$$

$$\cos \alpha = \frac{|(3, 1, -4) \cdot (-1, 3, 2)|}{\sqrt{26} \sqrt{14}} = \frac{8}{2\sqrt{91}}$$

**Distance from a point  $P : (x_1, y_1, z_1)$   
to a line  $\mathcal{L}$ :**

$$d(P, \mathcal{L}) = \frac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}.$$

## Example:

$$P : (1, 2, 3),$$

$$\mathcal{L} : x = 1 + t, \quad y = -2t, \quad z = 2 + 3t$$

$$\text{Answer: } \frac{\sqrt{69}}{\sqrt{14}}$$

## Planes: (12.7)

Given a point  $P$  and a vector  $\mathbf{N}$ .

There is one and only one plane  $\mathcal{P}$  which is perpendicular to  $\mathbf{N}$  and passes through  $P$ .

Given  $P : (x_0, y_0, z_0)$  and

$$\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}.$$

The set of all points  $Q : (x, y, z)$   
such that

$$\mathbf{N} \cdot \overrightarrow{PQ} = 0$$

is a plane  $\mathcal{P}$ .

$\mathbf{N}$  is a **normal** (vector) to  $\mathcal{P}$ .

1. **Vector equation:**  $\mathbf{N} \cdot \overrightarrow{PQ} = 0.$

2. **Standard Form equation:**

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

c.f. Point-slope equation for a straight line.

3. **General equation:**

$$Ax + By + Cz + D = 0 \quad \text{or}$$

$$Ax + By + Cz = E.$$

To find an equation for a plane, you need:

**1.** A point  $P$  which lies in the plane.

**2.** A normal vector  $\mathbf{N}$ .

## Examples:

1. Find an equation for the plane which is parallel to the plane

$$5x - 2y + 3z = 4$$

and passes through  $P : (2, 3, -1)$ .

2. Find an equation for the plane which is perpendicular to the line

$$\mathcal{L} : x = 1 + 6t, y = 2 - 2t, z = 4 + 5t$$

and passes through  $P : (-7, 4, 2)$ .

**3. a.** Show that the points

$$P : (3, -4, 1), Q : (4, 2, -1), R : (-1, 1, -2)$$

are not collinear.

**b.** Find an equation for the plane determined by  $P, Q, R$ .

**Intersections:** Given two planes  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Exactly one of the following holds:

1.  $\mathcal{P}_1, \mathcal{P}_2$  are parallel.
2.  $\mathcal{P}_1, \mathcal{P}_2$  coincide.
3.  $\mathcal{P}_1, \mathcal{P}_2$  intersect in a straight line  $\mathcal{L}$ .  $\mathbf{N}_1 \neq \lambda \mathbf{N}_2$

$\mathbf{d} = \mathbf{N}_1 \times \mathbf{N}_2$  is a direction vector for  $\mathcal{L}$ .

## Examples:

1. Determine whether the planes

$$\mathcal{P}_1 : -2x + 4y + 6z = 5$$

$$\mathcal{P}_2 : x - 2y - 3z - 3 = 0$$

are parallel, coincident, or intersect  
in a straight line

2. Determine whether the planes

$$\mathcal{P}_1 : (x + 1) - 2(y - 1) - 3(z - 2) = 0,$$

$$\mathcal{P}_2 : -2x + 4y + 6z = 18$$

are parallel, coincident, or intersect  
in a straight line

3. Prove that the planes

$$\mathcal{P}_1 : 4x + 4y - 2z = 9, \quad \mathcal{P}_2 : 2x + y + z = -3$$

intersect in a straight line and find parametric equations for the line of intersection.

**Angle between two planes:**

$$0 \leq \theta \leq \pi/2$$

$$\mathcal{P}_1 : A_1x + B_1y + C_1z = D_1,$$

$$\mathcal{P}_2 : A_2x + B_2y + C_2z = D_2$$

$$\mathbf{N}_1 = A_1 \mathbf{i} + B_1 \mathbf{j} + C_1 \mathbf{k}$$

$$\mathbf{N}_2 = A_2 \mathbf{i} + B_2 \mathbf{j} + C_2 \mathbf{k}$$

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|}$$

**Example:**

Find the cosine of the angle between

$$\mathcal{P}_1 : x + 2y + 3z = 2,$$

$$\mathcal{P}_2 : -3x + 4y + z = 4$$

## Distance from a point to a plane:

Given a plane

$$\mathcal{P} : Ax + By + Cz + D = 0$$

and a point  $P : (x_1, y_1, z_1)$  which does not lie on  $\mathcal{P}$ .

$$\begin{aligned}
d(P_1, \mathcal{P}) &= | \|\overrightarrow{P_0P_1}\| \cos \theta | \\
&= \frac{|\|\overrightarrow{P_0P_1}\| \|\mathbf{N}\| \cos \theta|}{\|\mathbf{N}\|} \\
&= \frac{|\overrightarrow{P_0P_1} \cdot \mathbf{N}|}{\|\mathbf{N}\|} \\
&= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}
\end{aligned}$$

## Example:

Show that the point  $P : (4, 1 - 3)$   
does not lie on the plane

$$\mathcal{P} : 2x + 3y - 4z + 27 = 0$$

and find the distance from  $P$  to  
 $\mathcal{P}$ .