Vector Functions; Calculus: (13.1, 13.2)

Given functions $f_1(t)$, $f_2(t)$, $f_3(t)$ defined on some t-interval I.

The vector

$$\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$$

$$= f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

is called *vector-valued function* or *vector function*.

Note: Same definition in 2-space and in n-space.

Examples:

1. $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$

$f_1(t) = 2+3t,\ f_2(t) = -1+4t,\ f_3(t) = 3-2t,$

$I:\ -\infty < t < \infty$

2. $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$

$f_1(t) = 3\sin 2t,\ f_2(t) = 4\cos 2t,\ f_3(t) = 0$

$I:\ 0 \le t \le 2\pi$

3. $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$

$f_1(t) = 4\cos 2t,\ f_2(t) = 4\sin 2t,\ f_3(t) = 2t$

$I:\ 0 \le t \le 4\pi$

NOTE: as t increases on I the tip of $\mathbf{f}$ traces out an oriented curve $\mathcal{C}$ in "space."

Calculus of Vector Functions

I. Limits: Let $c \in I$.

$$\lim_{t \to c} \mathbf{f}(t) = \mathbf{L} = L_1\,\mathbf{i} + L_2\,\mathbf{j} + L_3\,\mathbf{k}$$

if and only if

$$\lim_{t \to c} \|\mathbf{f}(t) - \mathbf{L}\| = 0$$

if and only if

$$\lim_{t \to c} f_1(t) = L_1,\ \lim_{t \to c} f_2(t) = L_2,\ \lim_{t \to c} f_3(t) = L_3$$

That is: limits are calculated "component-wise."

Example:

$$\mathbf{f}(t) = 2\cos\,t\,\mathbf{i} + 2\sin\,t\,\mathbf{j} + 2t\,\mathbf{k}$$

$$\lim_{t \to 2\pi/3} \mathbf{f}(t) = -\mathbf{i} + \sqrt{3}\,\mathbf{j} + 4\pi/3\,\mathbf{k}$$

Arithmetic of limits:

Let $\mathbf{f}$ and $\mathbf{g}$ be vector functions and let u be a scalar function.

Suppose $\mathbf{f}(t) \to \mathbf{L}, \quad \mathbf{g}(t) \to \mathbf{M} \quad u(t) \to A.$

1. $\mathbf{f}(t) + \mathbf{g}(t) \to \mathbf{L} + \mathbf{M}$

2. $\mathbf{f}(t) - \mathbf{g}(t) \to \mathbf{L} - \mathbf{M}$

3. $\mathbf{f}(t) \cdot \mathbf{g}(t) \to \mathbf{L} \cdot \mathbf{M}$

4. $\mathbf{f}(t) \times \mathbf{g}(t) \to \mathbf{L} \times \mathbf{M}$

5. $\alpha\mathbf{f}(t) \to \alpha\,\mathbf{L}$

6. $u(t)\mathbf{f}(t) \to A\,\mathbf{L}$

II. Continuity

$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$ is continuous at $t = c$ if and only if

$$\lim_{t \to c} \mathbf{f}(t) = \mathbf{f}(c).$$

$\mathbf{f}$ is continuous on an interval I if and only if it is continuous at each point $c \in I$.

$\mathbf{f}$ is a continuous vector function if and only if each of its components

$$f_1,\ f_2,\ f_3$$

is a continuous function.

III. Differentiation:

$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$ is differentiable at t if and only if

$$\lim_{h \to 0} \frac{1}{h}\left[\mathbf{f}(t+h) - \mathbf{f}(t)\right] \text{ exists.}$$

Notation: $\mathbf{f}'(t)$

$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$ is differentiable at t if and only if each of $f_1,\ f_2,\ f_3$ is differentiable at t and, if so,

$$\mathbf{f}'(t) = f_1'(t)\,\mathbf{i} + f_2'(t)\,\mathbf{j} + f_3'(t)\,\mathbf{k}$$

Example:

$\mathbf{f}(t) = e^{2t}\,\mathbf{i} + \cos 3t\,\mathbf{j} + \ln t\,\mathbf{k}$

$\mathbf{f}'(t) = 2e^{2t}\,\mathbf{i} - 3\sin 3t\,\mathbf{j} + \dfrac{1}{t}\,\mathbf{k}$

Integration: $\mathbf{f} = f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}$ is integrable if and only if each of $f_1,\ f_2,\ f_3$ is integrable and, if so,

$$\int \mathbf{f}(t)\,dt = \left(\int f_1(t)\,dt, \int f_2(t)\,dt, \int f_3(t)\,dt\right)$$

Example:

$\mathbf{f}(t) = e^{2t}\,\mathbf{i} + \cos\, 3t\,\mathbf{j} + \ln\, t\,\mathbf{k}$

$\int \mathbf{f}(t)\,dt =$

$\frac{1}{2}e^{2t}\,\mathbf{i} + \frac{1}{3}\sin\, 3t\,\mathbf{j} + (t\ln\, t - t)\,\mathbf{k} + \mathbf{C}$

Differentiation formulas: Suppose $\mathbf{f}$ and $\mathbf{g}$ are differentiable vector functions and u is a differentiable scalar function.

1. $(\mathbf{f} \pm \mathbf{g})'(t) = \mathbf{f}'(t) \pm \mathbf{g}'(t)$

2. $(\alpha \mathbf{f})'(t) = \alpha \mathbf{f}'(t)$ (α constant)

3. $(u\,\mathbf{f})'(t) = u(t)\,\mathbf{f}'(t) + u'(t)\mathbf{f}(t)$

4. $(\mathbf{f} \cdot \mathbf{g})'(t) = \mathbf{f}(t) \cdot \mathbf{g}'(t) + \mathbf{g}(t) \cdot \mathbf{f}'(t)$

5. $(\mathbf{f} \times \mathbf{g})'(t) = \mathbf{f}(t) \times \mathbf{g}'(t) + \mathbf{f}'(t) \times \mathbf{g}(t)$

   NOTE: the order in 5!!

6. $[\mathbf{f}(u(t)]' = \mathbf{f}'[u(t)]\,u'(t)$ the chain rule

Examples: If

$$\mathbf{f}(t) = 2t^2\,\mathbf{i} + (3t+1)\,\mathbf{j} + t^3\,\mathbf{k},$$

$$\mathbf{g}(t) = \mathbf{i} + t^2\,\mathbf{j} + 2t^4\,\mathbf{k},$$

$$u(t) = e^{2t},$$

then

- $(\mathbf{f}+\mathbf{g})'(t) = \mathbf{f}'(t) + \mathbf{g}'(t)$

  $$= 4t\,\mathbf{i} + (3+2t)\,\mathbf{j} + (3t^2+8t^3)\,\mathbf{k}$$

- $(\mathbf{f}\cdot\mathbf{g})'(t) = \mathbf{f}(t)\cdot\mathbf{g}'(t) + \mathbf{f}'(t)\cdot\mathbf{g}(t)$

  $$= 6t + 9t^2 + 14t^6$$

- $(\mathbf{f}[u(t)])' = \mathbf{f}'[u(t)]u'(t)$

  $$= 8e^{4t}\,\mathbf{i} + 6e^{2t}\,\mathbf{j} + 6e^{6t}\,\mathbf{k}$$

- **Important property:**

If $\mathbf{f}$ has constant norm, then

$$\mathbf{f} \perp \mathbf{f}'$$

Proof: Suppose $\|\mathbf{f}(t)\| \equiv C$, constant.

Curves; Tangent Vector, etc. (13.3)

Given

$$x = x(t),\ y = y(t),\ z = z(t)$$

differentiable functions on a t-interval I.

Let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

As t ranges over the interval I, the tip of $\mathbf{r}$ traces out a curve $\mathcal{C}$.

$\mathcal{C}$ is a *differentiable curve* parametrized by $\mathbf{r}$ with parameter t.

$\mathcal{C}$ is an *oriented* curve. The positive direction on I induces an orientation on $\mathcal{C}$.

Tangent vector:

Choose a point $c \in I$. The vector

$$\mathbf{r}'(c) = x'(c)\,\mathbf{i} + y'(c)\,\mathbf{j} + z'(c)\,\mathbf{k}$$

is a direction vector for the line tangent to $\mathcal{C}$ at the point $P : (x(c), y(c), z(c))$ on $\mathcal{C}$ (provided $\mathbf{r}(c) \neq \mathbf{0}$).

$\mathbf{r}'(c)$ points in the direction of increasing t.

Tangent line: The line through P with direction vector

$$\mathbf{r}'(c) = x'(c)\,\mathbf{i} + y'(c)\,\mathbf{j} + z'(c)\,\mathbf{k}$$

is tangent to $\mathcal{C}$ at P.

Examples:

1. Find a vector equation for the line tangent to

$$\mathbf{r}(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 3t\,\mathbf{k}$$

at the point where $t = \pi/3$.

2. Find parametric equations for the line tangent to

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$$

at the point $(6, 9, 9)$.

Unit tangent vector, principal normal vector:

Given

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

Assume $\|\mathbf{r}'(t)\| \neq 0$ for all t.

The vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

is called the *unit tangent vector*.

$\mathbf{T}$ and $\mathbf{r}'$ have the same direction.

Since $\mathbf{T}$ has constant norm

$$\mathbf{T} \perp \mathbf{T}'$$

(see slide 10)

The unit vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

is called the *principal normal vector*

Examples:

1. Find the unit tangent vector and the principal normal vector to

$$\mathbf{r}(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 3t\,\mathbf{k}$$

2. Find the unit tangent vector and the principal normal vector to

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$$

at the point where $t = 1$.

Let $P : (x_0, y_0, z_0)$ be a point on $\mathcal{C}$. The line through P with direction vector $\mathbf{N}$ is the *normal line* to $\mathcal{C}$ at P.

Example: Find parametric equations for the normal line to

$$\mathbf{r}(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 3t\,\mathbf{k}$$

at the point where $t = \pi/3$.

Osculating plane:

Fix a point P on the curve $\mathcal{C}$. The unit tangent and principal normal vectors determine a plane $\mathcal{P}$ called the *osculating plane* at P.

If $P : (x_0, y_0, z_0)$ is a point on $\mathcal{C}$ corresponding to $t = t_0$ and

$$\mathbf{T}(t_0) \times \mathbf{N}(t_0) = M_1\,\mathbf{i} + M_2\,\mathbf{j} + M_3\,\mathbf{k},$$

then

$$M_1(x - x_0) + M_2(y - y_0) + M_3(z - z_0) = 0$$

is an equation for the osculating plane.

Examples:

1. Find an equation for the osculating plane to the curve

$$\mathbf{r}(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j} + 3t\,\mathbf{k}$$

at the point where $t = \pi/3$.

2. Find an equation for the osculating plane to the curve

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}\,t^3\,\mathbf{k}$$

at the point where $t = 1$.

Arc Length (13.5)

Given

$$x = x(t),\ y = y(t),\ z = z(t), \quad I : a \le t \le b$$

differentiable functions on I.

Let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

$\mathbf{r}$ determines a differentiable, oriented curve $\mathcal{C}$. The *length* of $\mathcal{C}$ is given by

$$\mathcal{L} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Examples:

1. $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}, \quad 1 \le t \le 4$

2. $\mathbf{r}(t) = 3\sin\, t\,\mathbf{i} + 3\cos\, t\,\mathbf{j} + \frac{4}{3}t^{3/2}\,\mathbf{k},$

$0 \le t \le 4$

3. $\mathbf{r}(t) = \ln\, t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}, \quad 1 \le t \le e$

Special Cases:

1: A plane curve, parametric form:

$$x = x(t), \quad y = y(t), \quad a \le t \le b:$$

$$\mathcal{L} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

2. A plane curve, $y = f(x) \quad a \le x \le b$:

$$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

3. A polar curve, $r = g(\theta) \quad \alpha \le \theta \le \beta$:

$$\mathcal{L} = \int_\alpha^\beta \sqrt{[g(\theta)]^2 + [g'(\theta)]^2}\, d\theta$$

Examples:

1. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$

2. $\mathbf{r}(t) = 2\cos 2t\,\mathbf{i} + 2\sin 2t\,\mathbf{j}, \quad 0 \le t \le 2\pi$

3. $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}, \quad 0 \le x \le 3$

4. $r = 1 + \cos\theta, \quad 0 \le \theta \le \pi$

Distance and speed:

$$\mathcal{C}: \ \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \ \ a \le t \le b.$$

The *distance* traveled by an object moving along the curve $\mathcal{C}$ at time t is given by:

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du.$$

The rate of change of distance with respect to time is called the *speed*:

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

Curvature:

The *curvature* of a curve is a measure of the rate at which the curve is "curving." The curvature of $\mathcal{C}$ is the magnitude of the change of the unit tangent vector with respect to arc length.

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

Special cases – plane curves:

1. $C: \; y = f(x)$:

$$\kappa = \frac{|f''(t)|}{\left(1 + [f']^2\right)^{3/2}}$$

2. $C: \; x = x(t), \quad y = y(t)$:

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\left([x']^2 + [y']^2\right)^{3/2}}$$

Examples:

1. $f(x) = mx + b$; a straight line.

2. $x = a\cos t,\ y = a\sin t$; circle, centered at the origin, radius a.

3. $x = 4\cos t,\ y = 3\sin t$; an ellipse.

General case:

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt}$$

Examples:

1.

$$\mathbf{r}(t) = 2\cos\, 2t\,\mathbf{i} + 2\sin\, 2t\,\mathbf{j} + 3t\,\mathbf{k}$$

2.

$$\mathbf{r}(t) = (\cos\, t + t\sin\, t)\,\mathbf{i} + (\sin\, t - t\cos\, t)\,\mathbf{j} + \frac{\sqrt{3}}{2}t^2\,\mathbf{k}$$

"Mechanics"

Let

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \;\; t \in I$$

denote the position of an object at time t

The object moves along the curve $\mathcal{C}$.

Velocity: The rate of change of position with respect to time.

$$\mathbf{r}'(t) = \mathbf{v}(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$$

Speed: The magnitude of velocity.

$$\|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

NOTE: $\|\mathbf{r}'(t)\| = \dfrac{ds}{dt}$

Acceleration: The rate of change of velocity with respect to time.

$$\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k}$$

Examples: Find the velocity, speed and acceleration of the particle whose position at time t is:

1. $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$

2.

$$\mathbf{r}(t) = (\cos\, t + t \sin\, t)\,\mathbf{i} + (\sin\, t - t\cos\, t)\,\mathbf{j} + \frac{\sqrt{3}}{2}t^2\,\mathbf{k}$$

Tangential and normal components of acceleration:

The acceleration vector

$$\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)bfk$$

is a linear combination of the unit tangent and principal normal vectors; i.e., the acceleration vector lies in the osculating plane:

$$\mathbf{a}(t) = a_T\,\mathbf{T} + a_N\,\mathbf{N}$$

The coefficients a_T and a_N are called the *tangential* and *normal* components of acceleration.

a_T and a_N are given by:

$$a_T = \frac{d^2 s}{dt^2}, \qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2$$

Examples:

1. $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t\,\mathbf{k}$

$$\frac{ds}{dt} = \sqrt{3}\,e^t, \quad \kappa = \frac{\sqrt{2}}{3}e^{-t}$$

$$a_T = \sqrt{3}\,e^t, \quad a_N = \sqrt{2}e^t$$

$$\mathbf{a} = \sqrt{3}\,e^t\,\mathbf{T} + \sqrt{2}\,e^t\,\mathbf{N}$$

2. $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}$

Find tangential and normal components at the point where $t = 1$.

$$\frac{ds}{dt} = 3, \quad \kappa = \frac{2}{9}$$

$$a_T = 2, \quad a_N = 2$$

$$\mathbf{a} = 2\,\mathbf{T} + 2\,\mathbf{N}$$