THE GRADIENT

Given a function of several variables:

$$z = f(x, y) \qquad \text{or} \qquad w = F(x, y, z)$$

The gradient of f is the vector:

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = f_x\,\mathbf{i} + f_y\,\mathbf{j}$$

The gradient of F is the vector:

$$\nabla F = \frac{\partial F}{\partial x}\,\mathbf{i} + \frac{\partial F}{\partial y}\,\mathbf{j} + \frac{\partial F}{\partial z}\,\mathbf{k} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

Directional Derivatives

Given a function $z = f(x, y)$ and a unit vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$.

The **directional derivative** of f at the point (x_0, y_0) in the direction $\mathbf{u}$ is the number

$$
\begin{aligned}
f'_{\mathbf{u}}(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \mathbf{u} \\
&= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2
\end{aligned}
$$

Geometric interpretation:

Given a function $w = F(x, y, z)$ and a unit vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$.

The directional derivative of F at the point (x_0, y_0, z_0) in the direction $\mathbf{u}$ is the number

$$F'_{\mathbf{u}}(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) \cdot \mathbf{u} =$$

$$F_x(x_0, y_0, z_0)u_1 + F_y(x_0, y_0, z_0)u_2 + F_z(x_0, y_0, z_0)u_3$$

Examples

1. $f(x,y) = 2x^2y^3 - \dfrac{3y}{x}$

(a) Find the directional derivative of f at the point $(1,-1)$ in the direction of the vector $\mathbf{a} = 3\,\mathbf{i} + 4\,\mathbf{j}$.

(b) Find the directional derivative of f at the point $(1,-1)$ in the direction of the line $y = \frac{5}{2}x + 2$

2. $F(x,y,z) = xy^2 + yz^2 + zx^2$

Find the directional derivative of F at the point $(1,-1,2)$ toward the point $(2,3,4)$.

Maximum and minimum directional derivatives

Functions of two variables: $z = f(x, y)$

$$f'_{\mathbf{u}} = \nabla f(x_0, y_0) \cdot \mathbf{u} = \|\nabla f\| \cos \theta$$

where θ is the angle between ∇f and $\mathbf{u}$.

Since $|\cos \theta| \leq 1$,

$$\textbf{max. direc. deriv.} = \|\nabla f(x_0, y_0)\|$$

which occurs when $\mathbf{u}$ points in the direction of ∇f $(\theta = 0)$.

$$\textbf{min. direc. deriv.} = -\|\nabla f(x_0, y_0)\|$$

which occurs when $\mathbf{u}$ points in the direction opposite to the direction of ∇f $(\theta = \pi)$.

Functions of three variables: $w = F(x, y, z)$

$$F'_{\mathbf{u}} = \nabla F(x_0, y_0, z_0) \cdot \mathbf{u} = \|\nabla F\| \cos \theta$$

where θ is the angle between ∇F and $\mathbf{u}$.

Since $|\cos \theta| \leq 1$,

$$\textbf{max. direc. deriv.} = \|\nabla F(x_0, y_0, z_0)\|$$

which occurs when $\mathbf{u}$ points in the direction of ∇F $(\theta = 0)$.

$$\textbf{min. direc. deriv.} = -\|\nabla F(x_0, y_0, z_0)\|$$

which occurs when $\mathbf{u}$ points in the direction opposite to the direction of ∇F $(\theta = \pi)$.

Examples:

1. $f(x,y) = 2x - x^2y + 2y^2 - \frac{x}{y}.$

(a) Find the maximum directional derivative of f at the point $(2,1)$.

(b) Find a unit vector in the direction of the minimum directional derivative of f at the point $(1,-1)$.

2. $F(x,y,z) = 2x^2 + 3xyz - yz^2.$

Find the minimum directional derivative of F at the point $(2,-1,2)$.

Chain rules: Given $z = f(x, y)$.

If $x = x(t)$ and $y = y(t)$, then the composition

$$f[x(t), y(t)]$$

is a function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \mathbf{r}'(t).$$

If $x = x(s, t)$ and $y = y(s, t)$, then the composition

$$f[x(s, t), y(s, t)]$$

is a function of s and t and

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Examples:

1. Set $f(x,y) = 2x^2y + y^3$,

where $x(t) = t^2 + 1,\ y(t) = e^{2t}$.

$$F(t) = f[x(t), y(t)] = 2(t^2+1)^2e^{2t} + e^{6t}$$

Find $\dfrac{dF}{dt}$

2. Set $f(x,y) = 2x^2y + y^3$,

where $x(s,t) = s^2t,\ y(s,t) = te^{2s}$.

$$F[(s,t)] = f[x(s,t), y(s,t)] = 2s^4t^3e^{2s} + t^3e^{6s}$$

Find $\dfrac{\partial F}{\partial s}, \quad \dfrac{\partial F}{\partial t}$

Chain rules: Given $w = F(x, y, z)$.

If $x = x(t)$, $y = y(t)$ and $z = z(t)$, then the composition

$$F[x(t), y(t), z(t)]$$

is a function of t and

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \nabla F \cdot \mathbf{r}'(t).$$

If $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$, then the composition

$$F[x(s, t), y(s, t), z(s, t)]$$

is a function of s and t and

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}.$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}.$$

Examples:

1. Set $F(x,y,z) = 2x^3y^2 - \sin(xyz^2)$,

where $x(t) = 3t^2,\ y(t) = 2t+1,\ z(t) = t^3$.

$$\mathcal{F}(t) = F[x(t), y(t), z(t)]$$

$$= 54t^6(2t+1)^2 - \sin\left(6t^6 + 3t^5\right)$$

Find $\dfrac{d\mathcal{F}}{dt}$.

2. Set $F(x,y,z) = 2xy^2 - 3xyz^2$,

where $x(t) = s^2t,\ y(t) = 2st,\ z(t) = e^{st}$.

$$\mathcal{F}(s,t) = F[x(s,t), y(s,t), z(s,t)]$$

$$= 8s^4t^3 - 6s^3t^2e^{2st}$$

Find $\dfrac{\partial \mathcal{F}}{\partial s}, \quad \dfrac{\partial \mathcal{F}}{\partial t}$.

More chain rules:

If $x = x(s,t)$ and $y = y(s)$, then the composition

$$f[x(s,t), y(s)]$$

is a function of s and t and

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{dy}{ds}.$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}.$$

and so on ....

Examples

1. $f(x, y) = x^2 \cos xy - 2xy^2,$

$$x = 2s^2 t, \quad y = s^3 - 1.$$

(a) Calculate $\dfrac{\partial f}{\partial s}$.

(b) Calculate $\dfrac{\partial f}{\partial t}$.

2. $F(x, y, z) = 2xy + 3xyz - yz^2,$

$x = e^{2t} + 2,\ y = 2st^2,\ z = 2t(s^2 - 1)$

(a) Calculate $\dfrac{\partial F}{\partial s}$.

(b) Calculate $\dfrac{\partial F}{\partial t}$.

Direction of the gradient: Normal lines, tangent lines

Given $z = f(x, y)$.

Let $\mathcal{C}: \ f(x, y) = C$ be a level curve of f and let (x_0, y_0) be a point on $\mathcal{C}$.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = f_x\,\mathbf{i} + f_y\,\mathbf{j}$$

is perpendicular to $\mathcal{C}$ at (x_0, y_0).

Normal line to $\mathcal{C}$ at (x_0, y_0):

$$x = x_0 + f_x(x_0, y_0)t, \qquad y = y_0 + f_y(x_0, y_0)t$$

or

$$y - y_0 = \frac{f_y(x_0, y_0)}{f_x(x_0, y_0)}(x - x_0)$$

Tangent line to $\mathcal{C}$ at (x_0, y_0):

$$y - y_0 = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}(x - x_0)$$

Example:

$f(x,y) = \frac{1}{2}x^2 + 2xy + 2y^3$

A level curve: $\mathcal{C}: \ \frac{1}{2}x^2 + 2xy + 2y^3 = 8$

$(2,1)$ is a point on $\mathcal{C}$.

$\nabla f = (x + 2y)\,\mathbf{i} + (2x + 6y^2)\,\mathbf{j}$

$\nabla f(2,1) = 4\,\mathbf{i} + 10\,\mathbf{j}$ [normal vector at $(2,1)$]

Normal line:

$x = 2 + 4t, \ \ y = 1 + 10t$ or $y - 1 = \dfrac{5}{2}(x - 2)$

Tangent line:

$x = 2 - 10t, \ \ y = 1 + 4t$ or $y - 1 = -\dfrac{2}{5}(x - 2)$

Direction of the gradient: Normal lines, tangent planes

Given $w = F(x, y, z)$.

Let $\mathcal{S}: \; F(x, y, z) = C$ be a level surface of F and let $P : (x_0, y_0, z_0)$ be a point on $\mathcal{S}$.

$$\nabla F(P) = F_x(P)\,\mathbf{i} + F_y(P)\,\mathbf{j} + F_z(P)\,\mathbf{k}$$

is normal [perpendicular] to $\mathcal{S}$ at P.

Normal line to $\mathcal{S}$ at P:

$$x = x_0 + F_x(P)t, \;\; y = y_0 + F_y(P)t, \;\; z = z_0 + F_z(P)t$$

Tangent plane to $\mathcal{C}$ at (x_0, y_0):

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0$$

Example:

$F(x, y, z) = x^2y + 2y^3 - 3xyz$

A level surface: $\mathcal{S}: \; x^2y + 2y^3 - 3xyz = 6$

$(1, 2, 2)$ is a point on $\mathcal{S}$.

$\nabla F = (2xy - 3yz)\,\mathbf{i} + (x^2 + 6y^2 - 3xz)\,\mathbf{j} - 3xy\,\mathbf{k}$

$\nabla F(1, 2, 2) = -8\,\mathbf{i} + 19\,\mathbf{j} - 6\,\mathbf{k}$ [normal at (1,2,2)]

Normal line:

$$x = 1 - 8t, \quad y = 2 + 19t, \quad z = 2 - 6t$$

Tangent plane:

$$-8(x - 1) + 19(y - 2) - 6(z - 2) = 0$$

Important special case:

The surface $\mathcal{S}: \ z = f(x, y)$ is a level surface of

$$w = F(x, y, z) = f(x, y) - z$$

namely $\mathcal{S}: \ F(x, y, z) = 0$.

Let $P: (x_0, y_0, z_0)$, $[z_0 = f(x_0, y_0)]$, be a point on $\mathcal{S}$.

$$\mathbf{N} = f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - \mathbf{k}$$

is normal [perpendicular] to $\mathcal{S}$ at P.

Normal line to $\mathcal{S}$ at P:

$$x = x_0 + f_x(x_0, y_0)t, \ \ y = y_0 + f_y(x_0, y_0)t, \ \ z = z_0 - t$$

Tangent plane to $\mathcal{C}$ at (x_0, y_0):

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example:

$$\mathcal{S}: \ z = f(x,y) = \sqrt{1 - x^2 - y^2}$$

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ is a point on $\mathcal{S}$.

Normal vector to $\mathcal{S}$:

$$\mathbf{N} = -\frac{x}{\sqrt{1-x^2-y^2}}\,\mathbf{i} - \frac{y}{\sqrt{1-x^2-y^2}}\,\mathbf{j} - \mathbf{k}$$

At $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

$$\mathbf{N} = -\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} - \mathbf{k}$$

Normal line:

$$x = \frac{1}{2} - \frac{\sqrt{2}}{2}t, \quad y = \frac{1}{2} - \frac{\sqrt{2}}{2}t, \quad z = \frac{\sqrt{2}}{2} - t$$

Tangent plane:

$$-\frac{\sqrt{2}}{2}\left(x - \frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(y - \frac{1}{2}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

OPTIMIZATION

A. Local Extrema:

Definitions: Local maximum, local minimum.

Theorem: If the function $z = f(x, y)$ has a local extremum at (x_0, y_0), then either

$$\nabla f(x_0, y_0) = \mathbf{0}, \quad \text{that is} \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

or

$$\nabla f(x_0, y_0) \text{ does not exist,}$$

i.e. at least one of $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ fails to exist.

Definitions: Critical point, stationary point.

Examples:

1. $f(x,y) = x^4 - 4x^2 - 2y$

$$f_x = 4x^3 - 8x, \qquad f_y = -2.$$

No stationary points: $\nabla f((x,y) \neq \mathbf{0}$ for all (x,y) $[\partial f/\partial y = -2 \neq 0]$

2. $f(x,y) = 2x^2 - 3xy + y^2 - 2x + y + 4$

$$f_x = 4x - 3y - 2, \qquad f_y = -3x + 2y + 1.$$

Stationary point: $(-1,-2)$

3. $f(x,y) = 4xy - x^4 - 2y^2 + \frac{1}{16}$

$$f_x = 4y - 4x^3, \qquad f_y = 4x - 4y.$$

Stationary points: $(0,0), (1,1), (-1,-1)$

4. $f(x,y) = 8xy\,e^{-(x^2+y^2)}$

$$\partial f/\partial x = 8e^{-(x^2+y^2)}\left[y - 2x^2y\right],$$

$$\partial f/\partial y = 8e^{-(x^2+y^2)}\left[x - 2xy^2\right]$$

Stationary points:

$(0,0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$,

$(-1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$

SECOND PARTIALS TEST: Assume f has continuous second partials.

Let $\mathbf{x}_0 = (x_0, y_0)$ be a stationary point.

Set

$$A = f_{xx}(\mathbf{x}_0), \quad B = f_{xy}(\mathbf{x}_0), \quad C = f_{yy}(\mathbf{x}_0),$$

$$D = AC - B^2.$$

1. If $D < 0$, then $\mathbf{x}_0$ is a saddle point of f.

2. If $D > 0$, then f has:

a local minimum at $\mathbf{x}_0$ if $A > 0$,

a local maximum at $\mathbf{x}_0$ if $A < 0$.

3. If $D = 0$, then ???????.

Examples con't

2. $f_{xx} = 4,\ f_{xy} = -3,\ f_{yy} = 2;$

$$D = 9 - 8 = 1 > 0.$$

	A	B	C	D	
$(0,0)$	4	-3	2	$1 > 0$	saddle pt.

3. $f_{xx} = -12x^2,\ f_{xy} = 4,\ f_{yy} = -4.$

	A	B	C	D	
$(0,0)$	0	4	-4	$16 > 0$	saddle pt.
$(1,1)$	-12	4	-4	$-32 < 0$ $A < 0$	loc. max.
$(1,1)$	-12	4	-4	$-32 < 0$ $A < 0$	loc. max.

4.

$$f_{xx} = -16xye^{-(x^2+y^2)}\left[3-2x^2\right],$$

$$f_{xy} = 8e^{-(x^2+y^2)}\left[1-2x^2\right]\left[1-2y^2\right],$$

$$f_{yy} = -16xye^{-(x^2+y^2)}\left[3-2y^2\right].$$

	A	B	C	D
$(0,0)$	0	8	0	
$(1/\sqrt{2}, 1/\sqrt{2})$	$-16/e$	0	$-16/e$	
$(1/\sqrt{2}, -1/\sqrt{2})$	$16/e$	0	$16/e$	
$(-1/\sqrt{2}, 1/\sqrt{2})$	$16/e$	0	$16/e$	
$(-1/\sqrt{2}, -1/\sqrt{2})$	$-16/e$	0	$-16/e$	

B. Absolute Extrema:

Given $z = f(x, y)$, f defined on a domain D.

DEFINITIONS: Absolute max, absolute min, closed, bounded.

THEOREM: If D is closed and bounded, and if f is continuous, then f has an absolute maximum and an absolute minimum on D. The absolute extrema of f occur either at a critical point "inside" D, or on the boundary of D. (C.f. Section 4.4)

5. Find the absolute extrema of

$$f(x, y) = 2x^2 + x + y^2 - 2$$

on $\quad D = \{(x, y) : x^2 + y^2 \leq 4\}.$

Critical pts inside D : $\left(-\frac{1}{4}, 0\right)$

The boundary of D is: $x^2 + y^2 = 4$

"Critical pts" on the boundary of D :

$$(2, 0), \quad (-2, 0), \quad \left(-\frac{1}{2},\ \pm\frac{1}{2}\sqrt{15}\right).$$

Absolute min: $f(-\frac{1}{4}, 0) = -\frac{17}{8}$; absolute max: $f(2, 0) = 8.$

6. Find the absolute extrema of

$$f(x, y) = x^2 - 2xy + 2y^2 - 2x$$

on the closed triangular region bounded by the x-axis, the $x = 3$ and the line $y = 2x$.

Critical pts inside D : $(2, 1)$

"Critical pts" on the boundary of D :

$$(0, 0), \quad (1, 0), \quad (3, 0), \quad (3, 3/2),$$

$$(3, 6), \quad (1/5, 2/5).$$

Absolute min: $f(2, 1) = -2$,

absolute max: $f(3, 6) = 39$.

7. According to U.S. Postal Service regulations, the length plus the girth (perimeter of a cross-section) of a package cannot exceed 108 inches. What are the dimensions of the rectangular box of maximum volume that is acceptable for mailing? What is the maximum volume?

Maximize $V = xyz$ where

$$2x + 2y + z = 108.$$

That is: maximize

$V = xy(108 - 2x - 2y) = 108xy - 2x^2y - 2xy^2$

on the region in the first quadrant bounded by the coordinate axes and the line $x + y = 54$.

Ans: $x = y = 18, z = 36$; $V = 11,664$ in^3.