MULTIPLE INTEGRALS

A. Double integrals/Repeated Integrals

1. $f(x,y) = 3x^2 + 2y$ on Ω:

$$0 \le x \le 2,\ x^2 \le y \le x+2.$$

Calculate $\displaystyle\int\int_\Omega f(x,y)\,dxdy$.

$$\int\int_\Omega f(x,y)\,dxdy = \int_0^2 \int_{x^2}^{x+2} (3x^2 + 2y)\,dy\,dx = \frac{316}{15}$$

2. $f(x,y) = xy + 3$ on Ω:

$$0 \le y \le 2,\ y^2 \le x \le 2y.$$

Calculate $\displaystyle\int\int_\Omega f(x,y)\,dxdy$.

$$\int\int_\Omega f(x,y)\,dxdy = \int_0^2 \int_{y^2}^{2y} (xy+3)\,dx\,dy = \frac{20}{3}$$

3. Interchange order of integration in Example 2.

$$\int\int_{\Omega} f(x,y)\,dxdy = \int_0^2 \int_{y^2}^{2y} (xy+3)\,dx\,dy$$
$$= \int_0^4 \int_{x/2}^{\sqrt{x}} (xy+3)\,dy\,dx = \frac{20}{3}$$

4. Interchange order in Example 1.

$$\int_0^2 \int_{x^2}^{x+2} (3x^2+2y)\,dy\,dx =$$

$$\int_0^2 \int_0^{\sqrt{y}} (3x^2+2y)\,dx\,dy + \int_2^4 \int_{y-2}^{\sqrt{y}} (3x^2+2y)\,dx\,dy$$

5. $f(x,y) = e^{-y^2}$ on Ω the region bounded by the y-axis, the line $y = 1$, and the line $y = \frac{1}{2}x$. Calculate $\int\int_\Omega f(x,y)\,dxdy$.

$$\int_0^2 \int_{x/2}^1 e^{-y^2}\,dy\,dx = \int_0^1 \int_0^{2y} e^{-y^2}\,dx\,dy = 1 - \frac{1}{e}$$

6. Calculate $\int_0^2 \int_{x^2}^4 2x\,\cos(y^2)\,dy\,dx$ by interchanging the order of integration.

$$\begin{aligned} \int_0^2 \int_{x^2}^4 2x\,\cos(y^2)\,dy\,dx &= \int_0^4 \int_0^{\sqrt{y}} 2x\,\cos(y^2)\,dx\,dy \\ &= \frac{1}{2}\sin(16) \end{aligned}$$

7. Find the volume of the solid bounded by the coordinate planes and the plane

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$$

$$V=\int_0^2\int_0^{-\frac{3}{2}x+3} 4\left(1-\frac{x}{2}-\frac{y}{3}\right)\,dy\,dx$$

8. Find the volume of the solid bounded above by the paraboloid $z=x^2+y^2$, below by the x,y-plane and on the sides by the cylinder $x^2+y^2=4$.

$$V=4\int_0^2\int_0^{\sqrt{4-x^2}}(x^2+y^2)\,dy\,dx=\cdots$$

B. Double integrals; Polar Coordinates

Given $F = F(r,\theta)$ continuous on

$$\Gamma: \; a \le r \le b, \; \alpha \le \theta \le \beta.$$

- Partition:

$$a = r_0 < r_1 < r_2 < \cdots < r_n = b,$$

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots \theta_n = \beta.$$

- Area of a "polar rectangle:"

$$\frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{r_i + r_{i-1}}{2} \Delta r_i \Delta\theta_j$$

- Riemann Sum:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} F(r_i^*, \theta_i^*) \frac{r_i + r_{i-1}}{2} \Delta r_i \Delta\theta_j$$

$$\to \int\int_\Gamma F(r,\theta)\, r\, dr\, d\theta \to \int_\alpha^\beta \int_a^b F(r,\theta)\, r\, dr\, d\theta.$$

• Double integral of $F(r, \theta)$ over the polar region

$$\Omega: \ \alpha \leq \theta \leq \beta, \ \rho_1(\theta) \leq r \leq \rho_2(\theta):$$

$$\int\int_\Omega F(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{\rho_1(\theta}^{\rho_2(\theta)} F(r,\theta)\, r\, dr\, d\theta.$$

Examples: 1. Given $\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta$

a What is Γ?

b What is the integrand?

c Evaluate

d Express the double integral in rectangular coordinates.

2. Given $\int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$

a. Evaluate.

b Explain the result.

Double integrals in rectangular coordinates to double integrals in polar coordinates. "Signals:"

• Integrating over a circle, or part of a circle,

and

• integrand involves $x^2 + y^2$.

Examples:

3. Evaluate

$$\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2}\, dx\, dy$$

by changing to polar coordinates

$$\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2}\, dx\, dy =$$

$$\int_0^{\pi/4} \int_0^{1} r^3\, r\, dr\, d\theta = \frac{\pi}{20}$$

4. Find the volume of the solid bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the x, y-plane:

$$V = 4 \int_0^3 \int_0^{\sqrt{9-x^2}} \left(9 - x^2 - y^2\right)\, dx\, dy =$$

$$4 \int_0^{\pi/2} \int_0^3 \left(9 - r^2\right) r\, dr\, d\theta = \frac{81\pi}{2}$$

5. Derive the formula for the volume of a sphere of radius a.

$$V = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy =$$

$$2\int_{0}^{2\pi}\int_{0}^{a} \sqrt{a^2-r^2}\,r\,dr\,d\theta$$

6. Find the volume of the solid bounded above by $z = e^{-x^2-y^2}$, below by the x, y-plane, and on the sides by the cylinder $x^2 + y^2 = a^2$.

$$V = \int_{0}^{2\pi}\int_{0}^{a} e^{-r^2}\,r\,dr\,d\theta = \pi\left(1 - e^{-a^2}\right)$$

7. Find the volume of the solid bounded above the paraboloid $z = x^2 + y^2$, below by the x, y-plane, and on the sides by the cylinder $x^2 + y^2 = 2y$.

$$V = \int_0^{\pi} \int_0^{2\sin\theta} r^2\, r\, dr\, d\theta = \frac{3\pi}{2}$$

Some Applications of Double Integrals

1. Volume:

2. Area:

3. Mass:

4. Center of Mass:

Examples:

1. A plate occupies the region

$$\Omega: \ 0 \le x \le 1, \ x^2 \le y \le 1.$$

The mass density at a point (x, y) on the plate is given by $\lambda(x, y) = xy$. Find the mass and center of mass of the plate.

2. A plate has the form of a half disc

$$\Omega: \ -R \le x \le R, \ 0 \le y \le \sqrt{R^2 - x^2}.$$

The mass density at a point (x, y) on the plate is the distance to the curve boundary. Find the mass and center of mass of the plate.

C. Triple Integrals.

I. Integration Over a "Box":

Given $f = f(x, y, z)$ continuous on the rectangular "box"

$B: \; a_1 \le x \le b_1, \; a_2 \le y \le b_2, \; a_3 \le z \le b_3.$

• Partition into "little" boxes B_{ijk}

• Let $m_{ijk} = \min f$ on B_{ijk}; $M_{ijk} = \max f$ on B_{ijk}; let (x_i^*, y_j^*, z_k^*) be an arbitrary point in B_{ijk}.

• Lower sum: $\displaystyle \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{q} m_{ijk} \Delta x_i \Delta y_j \Delta z_k$

• Upper sum: $\displaystyle \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{q} M_{ijk} \Delta x_i \Delta y_j \Delta z_k$

• Riemann sum:

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{q} f(x_i^*, y_j^*, z_k^*)(\Delta x_i \Delta y_j \Delta z_k$$

• Limit as $n, p, q \to \infty$, mesh $\to 0$ exists and is denoted by

$$\int\int\int_B f(x, y, z)\, dxdydz$$

II. Integration Over an Arbitrary Solid:

$$\int\int\int_S f(x,y,z)\,dxdydz$$

III. Properties:

1. Linear

2. Order

3. Additivity

IV. Applications:

1. $\int\int\int_S f(x,y,z)\,dxdydz =$ "Volume" of a "hypersolid"

2. $\int\int\int_S 1\,dxdydz =$ volume of S.

V. Reduction to a Repeated Integral:

1. Type I Region:

$$a \le x \le b, \quad \phi_1(x) \le y \le \phi_2(x),$$

$$\psi_1(x,y) \le z \le \psi_2(x,y).$$

2. Type II Region:

$$c \le y \le d, \quad \phi_1(y) \le x \le \phi_2(y),$$

$$\psi_1(x,y) \le z \le \psi_2(x,y).$$

3. Etc. There are 6 regions in all.

Examples

1. Evaluate: $\displaystyle\int_0^1 \int_{1-x}^{1+x} \int_0^{xy} 4z\,dz\,dy\,dx$

Ans: $\dfrac{11}{9}$

2. Calculate $\displaystyle\int\int\int_T z\,dxdydz$ where T is the tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$.

Ans: $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \frac{1}{24}$

or: $\displaystyle\int_0^1 \int_0^{1-z} \int_0^{1-y-z} z\,dx\,dy\,dz = \frac{1}{24}.$

or .....

3. Find the volume of the solid bounded above by the cylindrical surface $x^2+z=4$, below by the plane $x+z=2$, and on the sides by the planes $y=0$ and $y=3$.

Ans: $\displaystyle\int_{-1}^{2}\int_{0}^{3}\int_{2-x}^{4-x^2} dz\,dy\,dx = \frac{27}{2}$

4. Find the volume of the solid bounded above by the plane $y + z = 2$, below by the x, y-plane, and on the sides by $x = 6$ and $y = \sqrt{x}$.

Ans: $V = \int_0^2 \int_{y^2}^6 \int_0^{2-y} dz\,dx\,dy = \frac{32}{3}$

$$V = \int_0^4 \int_0^{\sqrt{x}} \int_0^{y-2} dz\,dy\,dx + \int_4^6 \int_0^2 \int_0^{y-2} dz\,dy\,dx$$

$$V = \int_0^2 \int_0^{2-y} \int_{y^2}^6 dx\,dz\,dy$$

5. Integrate $f(x,y,z) = 2z$ over the solid S bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the x,y-plane, and on the sides by the planes $x = \sqrt{3}\,y$ and $y = \sqrt{3}\,x$.

Ans:

$$\int_0^{3/2} \int_{x/\sqrt{3}}^{x\sqrt{3}} \int_0^{9-x^2-y^2} 2z\,dz\,dy\,dx \;+$$

$$\int_{3/2}^{3\sqrt{3}/2} \int_{x/\sqrt{3}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} 2z\,dz\,dy\,dx$$

$$= \frac{81\pi}{4}$$

6. Find the volume of the solid bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the cone $z = \sqrt{3(x^2 + y^2)}$.

Ans: $V = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$

$= \dfrac{8\pi}{3} \left(2 - \sqrt{3}\right)$

I. Cylindrical Coordinates:

- $(x, y, z) \to (r, \theta, z)$ where

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z \qquad z = z$$

- **Triple Integrals**

$$\int\int\int_{\Omega} f(x, y, z)\, dxdydz \longrightarrow \int\int\int_{\Gamma} F(r, \theta, z)\, r\, dr\, d\theta\, dz$$

where $F(r, \theta, z) = f(r\cos\theta, r\sin\theta, z)$

- Use when there is an axis of symmetry; "signals:" integrand involves $x^2 + y^2$; integrating over a circle or part of a circle in the x, y-plane.

Examples:

1. Given: $\int_0^{\pi} \int_0^{2} \int_0^{4-r^2} r \, dz \, dr \, d\theta.$

(a) Sketch the solid determined by the limits.

(b) Evaluate the integral.

(c) Interpret the result.

$$\int_0^{\pi} \int_0^{2} \int_0^{4-r^2} r \, dz \, dr \, d\theta = 4\pi.$$

2. Convert

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

into a triple integral in cylindrical coordinates. Sketch the solid determined by the limits.

Ans: $\int_0^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} r \cdot r\, dz\, dr\, d\theta =$

$$\int_0^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} r^2\, dz\, dr\, d\theta$$

3. The cylinder $x^2 + y^2 = 4, \ z \geq 0$ is sliced by the plane $z = 4 + y$. Determine the volume of the "sliced" cylinder.

Ans: $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4+y} 1 \, dz \, dy \, dx =$

$$\int_0^{2\pi} \int_0^2 (4 + r \sin\theta) \, r \, dr \, d\theta = 16\pi$$

4. Draw the solid that is bounded above by a portion of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and below by the cone $z = \sqrt{3x^2 + 3y^2}$.

Set up a triple integral that gives the volume of the solid and then find its volume.

Ans:

$$4\int_0^{1/2}\int_0^{\sqrt{\frac{1}{4}-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx =$$

$$4\int_0^{\pi/2}\int_0^{1/2}\left(\sqrt{1-r^2} - r\sqrt{3}\right) r\,dr\,d\theta = \frac{\pi}{3}\left(2-\sqrt{3}\right).$$

II. Spherical Coordinates:

$(x, y, z) \to (\rho, \phi, \theta)$ where

$$x = \rho \sin\phi \cos\theta \qquad \rho^2 = x^2 + y^2 + z^2$$

$$y = \rho \sin\phi \sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = \rho \cos\phi \qquad \cos\phi = \frac{z}{\rho}$$

Triple Integrals

$$\int\int\int_\Omega f(x, y, z)\, dxdydz \longrightarrow$$

$$\int\int\int_\Gamma F(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

where

$$F(\rho, \phi, \theta)) =$$

$$f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta, \rho \cos\phi)$$

- Use when there is spherical symmetry;

"Signals:"

Integrand involves $x^2 + y^2 + z^2$;

Integrating over a sphere or part of a sphere.

Examples:

1. Given

$$\int\int\int_{\Omega} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

(a) what is the integrand?

(b) What is Ω?

(c) Evaluate the integral.

Ans: 8π

2. Give the value of

$$\int_0^{\pi} \int_0^{\pi/2} \int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

without calculating the integral.

Ans: 9π (1/4 the volume of a sphere of radius 3)

3. Evaluate

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

Ans:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy =$$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{3} \rho^4 \sin\phi \,\cos\phi \,d\rho\,d\phi\,d\theta = \frac{243\pi}{10}$$

4. Evaluate

$$\int_{-1/2}^{1/2}\int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$$

by changing to spherical coordinates.

Ans: $\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{\pi}{3}\left(2-\sqrt{3}\right).$