

# VECTOR FIELDS

Given a vector function

$$\mathbf{h}(x, y) = h_1(x, y) \mathbf{i} + h_2(x, y) \mathbf{j}$$

defined on some region  $\Omega$  in the plane. The set of vectors

$$\mathbf{h}(x, y), \quad (x, y) \in \Omega$$

is called a **vector field**.

Given a vector function  $\mathbf{h}(x, y, z) =$

$$h_1(x, y, z) \mathbf{i} + h_2(x, y, z) \mathbf{j} + h_3(x, y, z) \mathbf{k}$$

defined on some region  $\Omega$  in the space. The set of vectors

$$\mathbf{h}(x, y, z), \quad (x, y, z) \in \Omega$$

is called a **vector field**.

# LINE INTEGRALS

## A. Line Integrals/Work

Given  $\mathbf{h}(x, y, z) =$

$$h_1(x, y, z) \mathbf{i} + h_2(x, y, z) \mathbf{j} + h_3(x, y, z) \mathbf{k}$$

a vector function (vector field) defined on some region  $\Omega$  in space, and a smooth (or piecewise smooth) curve  $C$  in  $\Omega$ :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} + z(u) \mathbf{k}, \quad a \leq u \leq b.$$

Or, given

$$\mathbf{h}(x, y) = h_1(x, y) \mathbf{i} + h_2(x, y) \mathbf{j}$$

a vector function (vector field) defined on some region  $\Omega$  in the plane, and a smooth (or piecewise smooth) curve  $C$ :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b.$$

in  $\Omega$ .

**DEFINITION:** The **line integral**  
**of  $\mathbf{h}$  over the curve  $C$** , denoted  
 $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ , is the number given by

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{h}[\mathbf{r}(u)] \cdot \mathbf{r}'(u) du.$$

## Examples:

1. Set

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k}$$

and let  $C$  be the curve:

$$\mathbf{r}(u) = u \mathbf{i} + u^2 \mathbf{j} + u^3 \mathbf{k}, \quad 0 \leq u \leq 2.$$

Calculate  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ .

**Answer:**  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{668}{7}$

2. Set

$$\mathbf{h}(x, y) = (xy + 2y) \mathbf{i} + (2x + y) \mathbf{j}.$$

Calculate  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$  on:

(a)  $C_1$  : The parabola  $y = x^2$  from  
(1, 1) to (3, 9).

(b)  $C_2$  : The line segment from (1, 1)  
to (3, 9).

**Answer:**

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 112,$$

$$\int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{344}{3}.$$

**NOTE:** The value of the line integral of  $\mathbf{h}$  depends upon the path; different curves give different values.

3.  $\mathbf{h}$  as in Example 2. Calculate

$$\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

where

a.  $C_3 : e^u \mathbf{i} + e^{2u} \mathbf{j}, 0 \leq u \leq \ln 3.$

$C_1$  and  $C_3$  are different parametrizations of the parabola with the same orientation.

**Answer:**  $\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 112$

**b.**  $C_4 : u \mathbf{i} + (4u - 3) \mathbf{j}, 1 \leq u \leq 3.$

$C_2$  and  $C_4$  are different parametrizations of the line segment with the same orientation

**Answer:**  $\int_{C_4} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{344}{3}$

## PROPERTY 1.

The line integral

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

is invariant under order-preserving

parametrizations of  $C$ .

4.  $\mathbf{h}$  as in Example 2. Calculate

$$\int_{C_5} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

where

$$C_5 : (3 - 2u)\mathbf{i} + (9 - 8u)\mathbf{j}, \quad 0 \leq u \leq 1.$$

( $C_2$  and  $C_5$  are parametrizations of the line segment with the orientations reversed.)

**Answer:**  $\int_{C_5} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -\frac{344}{3}.$

## PROPERTY 2.

Reversing the direction of  $C$  changes the sign of

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

That is,

$$\int_{-C} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = - \int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

$$\text{C.f. } \int_b^a f(x) dx = - \int_a^b f(x) dx$$

5. Set

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + x^2 z \mathbf{j} + xyz \mathbf{k}.$$

Calculate  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$  over the paths

$C_1$ ,  $C_2$  and  $C_3$  where

(a)  $C_1$  is the line segment from  $(0, 0, 0)$  to  $(1, 1, 1)$ .

**Answer:**

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{5}{6},$$

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + x^2 z \mathbf{j} + xyz \mathbf{k}.$$

**(b)**  $C_2$  : the parabolic segment  $y = x^2$  from  $(0, 0, 0)$  to  $(1, 1, 0)$  followed by the line segment from  $(1, 1, 0)$  to  $(1, 1, 1)$ .

**Answer:**

$$\int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{3}{4},$$

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + x^2 z \mathbf{j} + xyz \mathbf{k}.$$

(c)  $C_3$ :

$$\mathbf{r}(u) = (1 - u) \mathbf{i} + (1 - u) \mathbf{j} + (1 - u) \mathbf{k},$$

$$0 \leq u \leq 1.$$

**Answer:**

$$\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -\frac{5}{6}.$$

## PROPERTY 3.

If a curve  $C$  is made up of connected smooth pieces  $C_1, C_2, \dots, C_n$ , then

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} =$$

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} + \dots + \int_{C_n} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

**Application:**  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  is the work done by the force  $\mathbf{F}$  acting on an object moving along the curve  $C$ :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} + z(u) \mathbf{k}, \quad a \leq u \leq b.$$

## Examples:

1. Find the work done by the force

$$\mathbf{F}(x, y) = (3x + 2xy) \mathbf{i} + (x^2 + 2y) \mathbf{j}$$

moving an object along the curve

$$\mathbf{r}(u) = (2u + 1) \mathbf{i} + u^3 \mathbf{j}, \quad 0 \leq u \leq 1$$

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2. The position of an object of mass  $m$  at time  $u$  is given by

$$\mathbf{r}(u) = 2u^2 \mathbf{i} + u^3 \mathbf{j} - u^4 \mathbf{k}, \quad 0 \leq u \leq 2.$$

**(a)** What is the force acting on the object?

**(b)** What is the work done by the force?

**Answer:**

**(a)** Remember:  $\mathbf{F} = m\mathbf{a}$ ; force equals mass  $\times$  acceleration

$$\mathbf{F}(u) = 4m \mathbf{i} + 6mu \mathbf{j} - 12mu^2 \mathbf{k}.$$

**(b)**  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 616 m.$

**B. The Fundamental Theorem for Line Integrals.** (NOTE: In this section we are considering only functions of two variables.)

**Example:** Set

$$\mathbf{h}(x, y) = (2xy - x^2) \mathbf{i} + (x^2 + y^2) \mathbf{j}.$$

Let  $C_1$  be the curve

$$\mathbf{r}(u) = u \mathbf{i} + (4 - u^2) \mathbf{j}, \quad -1 \leq u \leq 2.$$

Let  $C_2$  be the curve

$$\mathbf{r}(u) = u \mathbf{i} + (2 - u) \mathbf{j}, \quad -1 \leq u \leq 2.$$

Let  $C_3$  be the curve  $\gamma_1 \cup \gamma_2$  where

$$\gamma_1 = -\mathbf{i} + (3 - u) \mathbf{j}, \quad 0 \leq u \leq 3;$$

$$\gamma_2 = u \mathbf{i}, \quad -1 \leq u \leq 2.$$

Calculate :

$$\int_{C_1} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r},$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r},$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r}.$$

**Answer:**

$$\int_{C_1} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} = \int_{C_2} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} =$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} = -15.$$

Set

$$f(x, y) = x^2y - \frac{1}{3}x^3 + \frac{1}{3}y^3$$

and calculate  $f(2, 0) - f(-1, 3)$ .

**Answer:**  $f(2, 0) - f(-1, 3) = -15$

???????

# The Fundamental Theorem of Line Integrals

Given a vector function

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and a curve  $C : \mathbf{r}(u)$ .

If  $\mathbf{h}$  is the gradient of a function  $f$ , i.e., if  $\nabla f = \mathbf{h}$ , then

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A})$$

where  $\mathbf{A} = \mathbf{r}(a)$  and  $\mathbf{B} = \mathbf{r}(b)$ .

**Recall:** The vector function

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

is the gradient of some function  $f$

if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

## Examples:

1. Set  $\mathbf{h}(x, y) =$

$$(3x^2y^3 + 2x + 1)\mathbf{i} + (3x^3y^2 - 4y + \pi \cos \pi y)\mathbf{j}$$

and let  $C$  be the curve

$$\mathbf{r}(u) = (u^3 - 3u)\mathbf{i} + (2 - u^2)\mathbf{j}, \quad 0 \leq u \leq 2.$$

Calculate  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ .

**Answer:**  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -58$

2. Set

$$\mathbf{h}(x, y) = (e^{2y} - 2xy) \mathbf{i} + (2xe^{2y} - x^2 + 1) \mathbf{j}$$

and let  $C$  be the curve

$$\mathbf{r}(u) = (u+1) \mathbf{i} + \ln(u+1) \mathbf{j}, \quad 0 \leq u \leq 1.$$

Calculate  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ .

**Answer:**  $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 7 - 3 \ln 2$

**DEFINITION.** The curve

$$C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b,$$

is a **closed** curve if  $\mathbf{r}(a) = \mathbf{r}(b)$ .

**COROLLARY 1.** If  $\mathbf{h}$  is the

gradient of some function  $f$ , and

if the curve  $C : \mathbf{r}(u)$  is **closed**,

then

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

3. Set  $\mathbf{h}(x, y) =$

$$(2x \sin y - e^{2x} + 4) \mathbf{i} + (x^2 \cos y + 2e^{3y} - y^2) \mathbf{j}.$$

Let  $C$  be the curve

$$r^2 = 4 \sin 2\theta, \quad 0 \leq \theta \leq 2\pi.$$

( $C$  is a *lemniscate*.) Calculate

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

**Answer:**

4. Set  $\mathbf{h}(x, y) =$

$$(2xy^2 + 2 \ln x - 1) \mathbf{i} + (2x^2y - 2 \sin y^2) \mathbf{j}.$$

Let  $C$  be the boundary of the triangular region enclosed by the  $x$ -axis, the line  $x = 3$ , and the line  $y = 2x$ .

Calculate

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

**Answer:**

## **COROLLARY 2. (Independence**

**of Path)** If  $\mathbf{h}$  is the gradient of some function  $f$ , and if  $C_1$  and  $C_2$  are any two curves which begin at  $\mathbf{A} = \mathbf{r}(a)$  and end at  $\mathbf{B} = \mathbf{r}(b)$ , then

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

In contrast, see NOTE, slide 8.

## C. GREEN'S THEOREM

**NOTE:** This discussion is also restricted to functions of two variables.

### Line Integrals in Differential No-

**tation:** Given a vector function

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and a curve  $C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C P(x, y) dx + Q(x, y) dy.$$

## EXAMPLE:

$$\mathbf{h}(x, y) = x^2y \mathbf{i} + 2xy \mathbf{j};$$

$$\mathbf{r}(u) = u^2 \mathbf{i} + (1 - u^3) \mathbf{j}, \quad 0 \leq u \leq 2.$$

$$\mathbf{h}(\mathbf{r}) \cdot \mathbf{r}'(u) \, du =$$

$$(-2u^8 + 6u^7 + 2u^5 - 6u^4) \, du$$

$$x^2y \, dx + 2xy \, dy =$$

$$(-2u^8 + 6u^7 + 2u^5 - 6u^4) \, du.$$

**DEFINITION:** A curve

$$C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b$$

is a **closed curve** if  $\mathbf{r}(a) = \mathbf{r}(b)$ ;

$C$  is a **simple closed curve** if it is closed and  $\mathbf{r}(u_1) \neq \mathbf{r}(u_2)$  for all  $u_1, u_2 \in (a, b)$ ,  $u_1 \neq u_2$ , i.e.,  $C$  does not intersect itself except at the two endpoints.

A simple closed curve is called a **Jordan curve**; the positive direction is counterclockwise. The region enclosed by a Jordan curve is called a **Jordan region**.

**GREEN'S THEOREM:** Given

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and  $C$ , a simple closed curve oriented in the counterclockwise direction and enclosing the Jordan region  $\Omega$ . Then

$$\begin{aligned} \oint_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} &= \oint_C P(x, y) dx + Q(x, y) dy \\ &= \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \end{aligned}$$

## Examples:

1. Calculate

$$\oint_C x^2 y \, dx + 2xy \, dy$$

where  $C$  is the simple closed curve enclosing the region bounded by

$$y = x^2 \text{ and } y = \sqrt{x}.$$

**Method 1:** Let

$$C_1 : \mathbf{r}(u) = u \mathbf{i} + u^2 \mathbf{j}, \quad 0 \leq u \leq 1;$$

$$C_2 : \mathbf{r}(u) = u \mathbf{i} + \sqrt{u} \mathbf{j}, \quad 0 \leq u \leq 1.$$

Calculate:

$$\int_{C_1} P(x, y) dx + Q(x, y) dy$$

$$- \int_{C_2} P(x, y) dx + Q(x, y) dy.$$

**Method 2: GREEN'S THEO-**  
**REM:**

$$\oint_C x^2 y \, dx + 2xy \, dy = \iint_{\Omega} (2y - x^2) \, dx \, dy$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2y - x^2) \, dy \, dx = \frac{3}{14}.$$

2. Calculate  $\oint_C x^2y dx + 2xy dy$

where  $C$  is the square with vertices

$$(0, 0), (1, 0), (1, 1), (0, 1)$$

traversed counterclockwise.

**Method 1:** Calculate:

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}.$$

## Method 2: GREEN'S THEO-

REM:

$$\oint_C xy^2 dx + 2xy dy = \iint_{\Omega} (2y - x^2) dx dy$$

$$= \int_0^1 \int_0^1 (2y - x^2) dy dx = \frac{2}{3}.$$

3. Calculate

$$\oint_C xy \, dx + x^2 \, dy$$

where  $C$  is the triangle with vertices

$$(0, 0), (2, 2), (0, 2).$$

**Answer:**  $\frac{4}{3}$

4. Calculate

$$\oint_C \left( e^{\sin x} - y^3 \right) dx + \left( x^3 + \sqrt{y^4 + 1} \right) dy$$

where  $C$  is the circle

$$x^2 + y^2 = 4.$$

**Answer:**  $24\pi$

5. Calculate

$$\oint_C (3x^2y^2 + 2x) dx + (2x^3y - 4y + 1) dy$$

where  $C$  is the simple closed curve formed by the upper semi-circle of radius 2 and the line segment connecting  $(-2, 0)$  to  $(2, 0)$ .

**Answer:**      0

## Corollary to Green's Theorem:

Given

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}.$$

If  $C$  is a simple closed curve, and

if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

(i.e.,  $\mathbf{h}$  is a gradient) then

$$\oint_C P(x, y) dx + Q(x, y) dy = 0$$

## Area of $\Omega$ using Green's Theorem:

If  $C$  a simple closed curve enclosing the region  $\Omega$ , then

$$\begin{aligned}\text{Area of } \Omega &= \int \int_{\Omega} 1 \, dx \, dy \\ &= \oint_C -y \, dx = \oint_C x \, dy \\ &= \frac{1}{2} \oint_C -y \, dx + x \, dy.\end{aligned}$$

**6.** Derive the formula for the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Ellipse:

$$\mathbf{r}(u) = a \cos u \mathbf{i} + b \sin u \mathbf{j}, \quad 0 \leq u \leq 2\pi$$