

LINE INTEGRALS

A. Line Integrals/Work

Given

$$\mathbf{h}(x, y, z) = h_1(x, y, z) \mathbf{i} + h_2(x, y, z) \mathbf{j} + h_3(x, y, z) \mathbf{k}$$

a vector function (vector field) defined on
some region Ω in space,

and a smooth (or piecewise smooth) curve
 C :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j} + z(u) \mathbf{k}, \quad a \leq u \leq b,$$

in Ω .

Or, given

$$\mathbf{h}(x, y) = h_1(x, y) \mathbf{i} + h_2(x, y) \mathbf{j}$$

a vector function (vector field) defined on some region Ω in the plane,

and a smooth (or piecewise smooth) curve C :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b.$$

in Ω .

DEFINITION: The **line integral of \mathbf{h} over the curve C** , denoted $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$, is the number given by

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{h}[\mathbf{r}(u)] \cdot \mathbf{r}'(u) du.$$

Examples:

1. Set

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k}$$

and let C be the curve:

$$\mathbf{r}(u) = u \mathbf{i} + u^2 \mathbf{j} + u^3 \mathbf{k}, \quad 0 \leq u \leq 2.$$

Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$.

Answer: $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{668}{7}$

2. Set

$$\mathbf{h}(x, y) = (xy + 2y) \mathbf{i} + (2x + y) \mathbf{j}.$$

Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ on:

(a) C_1 : The parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$.

(b) C_2 : The line segment from $(1, 1)$ to $(3, 9)$.

Answer: $\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 112,$

$$\int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{344}{3}.$$

NOTE: The value of the line integral of \mathbf{h} depends upon the path; different curves give different values.

3. \mathbf{h} as in Example 2. Calculate

$$\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

where

(a) $C_3 : e^u \mathbf{i} + e^{2u} \mathbf{j}, 0 \leq u \leq \ln 3.$

(b) $C_4 : u \mathbf{i} + (4u - 3) \mathbf{j}, 1 \leq u \leq 3.$

C_1 and C_3 are different parametrizations of the parabola with the same orientation; C_2 and C_4 are different parametrizations of the line segment with the same orientation

Answer:

$$\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 112; \quad \int_{C_4} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{344}{3}$$

PROPERTY 1.

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

is invariant under order-preserving parametrizations of C .

4. \mathbf{h} as in Example 2. Calculate

$$\int_{C_5} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$$

where

$$C_5 : (3 - 2u)\mathbf{i} + (9 - 8u)\mathbf{j}, \quad 0 \leq u \leq 1.$$

(C_2 and C_5 are parametrizations of the line segment with the orientations reversed.)

Answer: $\int_{C_5} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -\frac{344}{3}.$

PROPERTY 2. Reversing the direction of C changes the sign of

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

That is,

$$\int_{-C} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = - \int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

5. Set

$$\mathbf{h}(x, y, z) = xy \mathbf{i} + x^2 z \mathbf{j} + xyz \mathbf{k}.$$

Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ over the paths C_1 , C_2 and C_3 where

(a) C_1 is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

(b) C_2 : the parabolic segment $y = x^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment from $(1, 1, 0)$ to $1, 1, 1$.

(c) C_3 :

$$\mathbf{r}(u) = (1-u) \mathbf{i} + (1-u) \mathbf{j} + (1-u) \mathbf{k}, \quad 0 \leq u \leq 1.$$

Answer:

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{5}{6},$$

$$\int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \frac{3}{4},$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -\frac{5}{6}.$$

PROPERTY 3. If a curve C is made up of connected smooth pieces C_1, C_2, \dots, C_n , then

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} + \dots \\ + \int_{C_n} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

Application: $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is the work done by the force \mathbf{F} acting on an object moving along the curve

$$C : \mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}, \quad a \leq u \leq b.$$

Example: The position of an object of mass m at time u is given by

$$\mathbf{r}(u) = 2u^2\mathbf{i} + u^3\mathbf{j} - u^4\mathbf{k}, \quad 0 \leq u \leq 2.$$

(a) What is the force acting on the object?

(b) What is the work done by the force?

Answer: (a) $\mathbf{F}(u) = 4m\mathbf{i} + 6mu\mathbf{j} - 12mu^2\mathbf{k}$

(b) $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 616m.$

B. The Fundamental Theorem for Line Integrals

NOTE: In this section we are considering only functions of two variables.

Example: Set

$$\mathbf{h}(x, y) = (2xy - x^2) \mathbf{i} + (x^2 + y^2) \mathbf{j}.$$

Let

$$C_1 : \mathbf{r}(u) = u \mathbf{i} + (4 - u^2) \mathbf{j}, \quad -1 \leq u \leq 2.$$

$$C_2 : \mathbf{r}(u) = u \mathbf{i} + (2 - u) \mathbf{j}, \quad -1 \leq u \leq 2.$$

$$C_3 = \gamma_1 \cup \gamma_2 \quad \text{where}$$

$$\gamma_1 = -\mathbf{i} + (3 - u) \mathbf{j}, \quad 0 \leq u \leq 3;$$

$$\gamma_2 = u \mathbf{i}, \quad -1 \leq u \leq 2.$$

Calculate :

$$\int_{C_1} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r},$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r},$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r}.$$

Answer:

$$\int_{C_1} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} = \int_{C_2} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} =$$

$$\int_{C_3} \mathbf{h}(\mathbf{r}(u)) \cdot d\mathbf{r} = -15.$$

Set

$$f(x, y) = x^2y - \frac{1}{3}x^3 + \frac{1}{3}y^3$$

and calculate $f(2, 0) - f(-1, 3)$.

Answer: $f(2, 0) - f(-1, 3) = -15$???????

The Fundamental Theorem of Line Integrals:

Given a vector function \mathbf{h} and a curve $C : \mathbf{r}(u)$. If \mathbf{h} is the gradient of a function f , i.e., if $\nabla f = \mathbf{h}$, then where $\mathbf{A} = \mathbf{r}(a)$ and $\mathbf{B} = \mathbf{r}(b)$.

Recall: The vector function

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

is the gradient of some function f if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Examples:

1. Set

$$\mathbf{h}(x, y) = (3x^2y^3 + 2x + 1)\mathbf{i} + (3x^3y^2 - 4y + \pi \cos \pi y)\mathbf{j}$$

and let C be the curve

$$\mathbf{r}(u) = (u^3 - 3u)\mathbf{i} + (2 - u^2)\mathbf{j}, \quad 0 \leq u \leq 2.$$

Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$.

Answer: $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = -58$

2. Set

$$\mathbf{h}(x, y) = (e^{2y} - 2xy) \mathbf{i} + (2xe^{2y} - x^2 + 1) \mathbf{j}$$

and let C be the curve

$$\mathbf{r}(u) = (u + 1) \mathbf{i} + \ln(u + 1) \mathbf{j}, \quad 0 \leq u \leq 1.$$

Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$.

Answer: $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 7 - 3 \ln 2$

DEFINITION. The curve

$$C : \mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j}, \quad a \leq u \leq b,$$

is a **closed** curve if $\mathbf{r}(a) = \mathbf{r}(b)$.

COROLLARY 1. If \mathbf{h} is the gradient of some function f , and if the curve $C : \mathbf{r}(u)$ is closed, then

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

COROLLARY 2. (Independence of Path)

If \mathbf{h} is the gradient of some function f , and if C_1 and C_2 are any two curves which begin at $\mathbf{A} = \mathbf{r}(a)$ and end at $\mathbf{B} = \mathbf{r}(b)$, then

$$\int_{C_1} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

See NOTE, slide 4.

3. Set

$$\mathbf{h}(x, y) = (2x \sin y - e^{2x} + 4) \mathbf{i} + (x^2 \cos y + 2e^{3y} - y^2) \mathbf{j}.$$

Let C be the curve

$$C : r^2 = 4 \sin 2\theta \quad 0 \leq \theta \leq 2\pi.$$

(C is a *lemniscate*.) Calculate

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}.$$

Answer:

C. GREEN'S THEOREM

NOTE: This discussion is restricted to functions of two variables.

Line Integrals in Differential Notation:

Given a vector function

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and a curve C :

$$\mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b.$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C P(x, y) dx + Q(x, y) dy.$$

EXAMPLE:

$$\mathbf{h}(x, y) = x^2y \mathbf{i} + 2xy \mathbf{j};$$

$$\mathbf{r}(u) = u^2 \mathbf{i} + (1 - u^3) \mathbf{j}, \quad 0 \leq u \leq 2.$$

$$\mathbf{h}(\mathbf{r}) \cdot \mathbf{r}'(u) du = (-2u^8 + 6u^7 + 2u^5 - 6u^4) du$$

$$x^2y dx + 2xy dy = (-2u^8 + 6u^7 + 2u^5 - 6u^4) du.$$

DEFINITION: A curve

$$C : \mathbf{r}(u) = x(u) \mathbf{i} + y(u) \mathbf{j}, \quad a \leq u \leq b$$

is a **closed curve** if $\mathbf{r}(a) = \mathbf{r}(b)$;

C is a **simple closed curve** if it is closed and $\mathbf{r}(u_1) \neq \mathbf{r}(u_2)$ for all $u_1, u_2 \in (a, b)$, $u_1 \neq u_2$, i.e., C does not intersect itself except at the two endpoints. A simple closed curve is called a **Jordan curve**; the positive direction is counterclockwise. The region enclosed by a Jordan curve is called a **Jordan region**.

GREEN'S THEOREM: Given

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

and C , a simple closed curve oriented in the counterclockwise direction and enclosing the Jordan region Ω . Then

$$\begin{aligned} \oint_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} &= \oint_C P(x, y) dx + Q(x, y) dy \\ &= \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \end{aligned}$$

Examples:

1. Calculate $\oint_C x^2y dx + 2xy dy$ where C is the simple closed curve enclosing the region bounded by $y = x^2$ and $y = \sqrt{x}$.

Method 1: Let

$$C_1 : \mathbf{r}(u) = u \mathbf{i} + u^2 \mathbf{j}, \quad 0 \leq u \leq 1;$$

$$C_2 : \mathbf{r}(u) = u \mathbf{i} + \sqrt{u} \mathbf{j}, \quad 0 \leq u \leq 1.$$

Calculate the line integrals:

$$\int_{C_1} P(x, y) dx + Q(x, y) dy - \int_{C_2} P(x, y) dx + Q(x, y) dy.$$

Method 2: Green's theorem:

$$\oint_C x^2y dx + 2xy dy = \int \int_{\Omega} (2y - x^2) dx dy =$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (2y - x^2) dy dx = \frac{3}{14}.$$

2. Calculate $\oint_C x^2 y dx + 2xy dy$ where C is the square with vertices

$$(0, 0), (1, 0), (1, 1), (0, 1)$$

traversed counterclockwise.

Method 1: Calculate the line integrals:

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}.$$

Method 2: Green's theorem:

$$\oint_C x^2 dx + 2xy dy = \int \int_{\Omega} (2y - x^2) dx dy =$$

$$\int_0^1 \int_0^1 (2y - x^2) dy dx = \frac{2}{3}.$$

3. Calculate

$$\oint_C xy \, dx + x^2 \, dy$$

where C is the triangle with vertices

$$(0, 0), (2, 2), (0, 2).$$

Answer: $\frac{4}{3}$

4. Calculate

$$\oint_C (e^{\sin x} - y^3) \, dx + (x^3 + \sqrt{y^4 + 1}) \, dy$$

where C is the circle

$$x^2 + y^2 = 4.$$

Answer: 24π

5. Calculate

$$\oint_C (3x^2y^2 + 2x) dx + (2x^3y - 4y + 1) dy$$

where C is the simple closed curve formed by the upper semi-circle of radius 2 and the line segment connecting $(-2, 0)$ to $(2, 0)$.

Answer: 0

Corollary to Green's Theorem: Given

$$\mathbf{h}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}.$$

If C is a simple closed curve, and if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

(i.e., \mathbf{h} is a gradient) then

$$\oint_C P(x, y) dx + Q(x, y) dy = 0$$

Area of Ω using Green's Theorem:

If C a simple closed curve enclosing the region Ω , then

$$\begin{aligned}\text{Area of } \Omega &= \int \int_{\Omega} 1 \, dx \, dy = \oint_C -y \, dx = \oint_C x \, dy \\ &= \frac{1}{2} \oint_C -y \, dx + x \, dy.\end{aligned}$$

6. Derive the formula for the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$