Solutions, Assignment #10

Section 7.1

1. Answer: The vectors $w_1 = \frac{1}{\sqrt{3}}(1, 1, -1)$ and $w_2 = \frac{1}{\sqrt{2}}(0, 1, 1)$ form an orthonormal basis for the solution set.

SOLUTION Find one vector which is a solution to the equation, for example $(1, 1, -1)$. Then, divide the vector by its length, obtaining the unit vector w_1. By inspection, find a vector v_2 which satisfies both the given equation and $w_1 \cdot v_2 = 0$. Then set $w_2 = \frac{1}{||v_2||} v_2$.

Section 7.2

1. Answer: The vectors $v_1 = \frac{1}{5}(3, 4)$ and $v_2 = \frac{1}{5}(-4, 3)$ form an orthonormal basis for $\mathbb{R}^2$.

SOLUTION Find these vectors using Gram-Schmidt orthonormalization (Theorem **??**). More specifically, calculate v_1 and v_2 such that:

$$\begin{aligned} v_1 &= \frac{1}{||w_1||} w_1 = \frac{1}{5}(3, 4). \\ v_2' &= w_2 - (w_2 \cdot v_1) v_1 = (1, 5) - \frac{23}{25}(3, 4) = \frac{1}{25}(-44, 33). \\ v_2 &= \frac{1}{||v_2'||} v_2' = \frac{5}{11}\left(\frac{1}{25}(-44, 33)\right) = \frac{1}{5}(-4, 3). \end{aligned}$$

2. Answer: The vectors $v_1 = \frac{1}{\sqrt{14}}(1, 2, 3)$ and $v_2 = \frac{1}{\sqrt{4746}}(19, 52, -41)$ form an orthonormal basis for W.

SOLUTION Use Gram-Schmidt orthonormalization (Theorem **??**) to calculate v_1 and v_2 such that:

$$\begin{aligned} v_1 &= \frac{1}{||w_1||} w_1 = \frac{1}{\sqrt{14}}(1, 2, 3). \\ v_2' &= w_2 - (w_2 \cdot v_1) v_1 = (2, 5, -1) - \frac{9}{14}(1, 2, 3) = \frac{1}{14}(19, 52, -41). \\ v_2 &= \frac{1}{||v_2'||} v_2' = \frac{14}{\sqrt{4746}}\left(\frac{1}{14}(19, 52, -41)\right) = \frac{1}{\sqrt{4746}}(19, 52, -41). \end{aligned}$$

Section 7.4

2. Answer: The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$, with respective eigenvectors $v_1 = (2, 1)$ and $v_2 = (1, -2)$.

SOLUTION Indeed, $v_1 \cdot v_2 = (2,1) \cdot (1,-2) = 0$, so the eigenvectors are orthogonal.

3. **Answer:** The eigenvectors of A are $w_1 = (1,1)$ and $w_2 = (1,-1)$ with eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$.

SOLUTION Note that any matrix of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ has eigenvectors w_1 and w_2 with eigenvalues $\lambda_1 = a+b$ and $\lambda_2 = a-b$. By iterating using `map`, we see that v_j approaches a multiple of $(1,1)$ as j increases for $v_0 \neq (1,-1)$. If v_0 is a multiple of $(1,-1)$, then v_j is a multiple of $(1,-1)$ for all j.

In Exercises **??** – **??** let u_1 be the eigenvector of matrix associated to the eigenvalue λ_1 where $|\lambda_1| > |\lambda_2|$. These exercises demonstrate that v_j approaches the direction of u_1 as j increases when v_0 is not a scalar multiple of u_2.

6. In this case, for any vector $v_0 = (x,y)$, the result of the iteration is $v_1 = Av_0 = 2(y,x)$. That is, any vector multiplied by A is reflected across the line $y = x$ and doubled in length. The result is different from that in Exercises **??** – **??** because the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, so $|\lambda_1| = |\lambda_2|$.

Section 8.1

1. (a) The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -3$, with corresponding eigenvectors $v_1 = (1,1)^t$ and $v_2 = (1,-1)^t$, respectively.

(b) Let

$$S = (v_1|v_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $D = S^{-1}AS = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$ is a diagonal matrix.

2. The eigenvectors of A are $v_1 = (1,1,-1)^t$ associated to eigenvalue $\lambda_1 = 2$; $v_2 = (1,-1,-1)^t$ associated to eigenvalue $\lambda_2 = -2$; and $v_3 = (1,-1,1)^t$ associated to eigenvalue $\lambda_3 = -4$. Find these vectors by solving $(A - \lambda I_3)v = 0$ for each eigenvalue λ. The matrix S such that $S^{-1}AS$ is diagonal is

$$S = (v_1|v_2|v_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

3. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenvector associated to λ_1 is $v_1 = (1,1,1)^t$. There are two eigenvectors associated to λ_2: $v_2 = (1,0,0)^t$

and $v_3 = (0, 1, 2)^t$.

$$S = (v_1|v_2|v_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

4. (a) Let $B = P^{-1}AP$ be a matrix similar to some invertible matrix A. Then

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P.$$

Since A^{-1} exists, B^{-1} exists also.

(b) If $B = P^{-1}AP$, then $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$. Therefore,

$$B + B^{-1} = P^{-1}AP + P^{-1}A^{-1}P = P^{-1}(A + A^{-1})P$$

since matrix multiplication is associative. Therefore, $A + A^{-1}$ is similar to $B + B^{-1}$.

5. Let $A = P^{-1}BP$ for some invertible matrix P, and let $D = S^{-1}AS$, where D is a diagonal matrix. Then

$$D = S^{-1}AS = S^{-1}(P^{-1}BP)S = (S^{-1}P^{-1})B(PS) = (PS)^{-1}B(PS).$$

Therefore, B is also similar to D, so B is real diagonalizable.

6. Let S be a matrix such that $D = S^{-1}AS$ is a diagonal matrix. Then

$$S^{-1}(A + \alpha I_n)S = S^{-1}AS + S^{-1}(\alpha I_n)S = D + \alpha I_n.$$

The matrices D and αI_n are both diagonal; so $D + \alpha I_n$ is also diagonal. Therefore, $A + \alpha I_n$ is diagonalizable.

9. Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is diagonal. The diagonal entries of $S^{-1}AS$ are the eigenvalues of A; that is, the diagonal entries equal ± 1. Therefore, $(S^{-1}AS)^2 = I_n$. But $(S^{-1}AS)^2 = S^{-1}A^2S$. Therefore, $S^{-1}A^2S = I_n$ which implies that $A^2 = I_n$.

10. Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is diagonal. The diagonal entries of $S^{-1}AS$ are the eigenvalues of A; that is, the diagonal entries equal 0 and 1. Therefore, $(S^{-1}AS)^2 = S^{-1}AS$. But $(S^{-1}AS)^2 = S^{-1}A^2S$. Therefore, $S^{-1}A^2S = S^{-1}AS$ which implies that $A^2 = A$.