Solutions, Assignment #2

Section 2.1

2. $(x, y, z) = (-\frac{2}{3}, -1, 3)$.

4. The matrices for the three systems are,

$$\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -2 & 1 \\ 3 & 3 \end{pmatrix}.$$

5. Answer: The system in part (a) has an infinite number of solutions, whereas the system in part (b) has no solution.

SOLUTION (a) Replace x in the second and third equations with $4 + y$ to obtain $4y - 2z = -10$ and $6y - 3z = -15$. Since these equations have identical solutions, the system can be restated as

$$\begin{aligned} x &= y + 4 \\ z &= -2y + 5 \end{aligned}$$

So, for each choice of y, there exists a single solution. For example, if $y = 1$, then $x = 5$ and $z = 3$; or if $y = 0$, then $x = 4$ and $z = 5$.

(b) If we again substitute for x using the first equation, the second and third equations become

$$2y - 3z = -2 \quad \text{and} \quad 2y - 3z = -4$$

These two expressions contradict each other, so there is no solution for this system.

6. Dick is 17 and Jane is 9.

9. Type `A(2,1) = -5`. MATLAB responds with

```
A =
     -1      1      2
     -5      1      2
```

10. Type `b(3) = 13` to obtain

```
b =
      1
      2
     13
```

11. Answer: According to MATLAB,

```
ans =
  -12.0495
   -0.8889
    7.8384
```

SOLUTION Write the system as $Ax = b$, where:

```
A =                                                  b =
    2.0000   -4.5000    3.1000                          4.2000
    1.0000    1.0000    1.0000                         -5.1000
    1.0000   -6.2000    1.0000                          1.3000
```

then type `A\b` to solve.

Section 2.2

1. Answer: The equation of the desired plane is $2x + 3y + z = -5$.

SOLUTION Note that a vector perpendicular to a plane is orthogonal to any vector connecting two points in the plane. So, if $X_0 = (-1, -2, 3)$ is one point in the plane perpendicular to $N = (2, 3, 1)$ and $X = (x, y, z)$ is any other point in that plane, then $(X - X_0) \cdot N = 0$. Substituting into this formula, we obtain

$$2(x+1) + 3(y+2) + 1(z-3) = 0 \quad \text{or} \quad 2x + 3y + z = -5.$$

3. Answer: The equation for the plane is $z = x$.

SOLUTION Note that the plane goes through the origin and contains the vectors $(1, 0, 1)$ and $(2, -1, 2)$, and therefore contains the points $(0, 0, 0)$, $(1, 0, 1)$, and $(2, -1, 2)$. The general equation for a plane is $ax + by + cz = d$. We can substitute the coordinates of the three points into this equation to get the linear system

$$\begin{array}{rcrcrcl} & & & & 0 & = & d \\ a & & & + & c & = & d \\ 2a & - & b & + & 2c & = & d \end{array}$$

We can solve the system by substitution to get $b = d = 0$ and $a = -c$, which yields the equation of the plane.

4. Answer: One such system is:

$$\begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ 2x & + & y & - & 4z & = & 0 \end{array}$$

SOLUTION The solution set contains all multiples of the vector $(1,2,1)$, so it contains the origin, since $0(1,2,1) = (0,0,0)$. The equation of any plane containing the origin is $ax+by+cz=0$. Substituting the point $(1,2,1)$ implies that $a+2b+c=0$. Any two equations which satisfy that condition and are not multiples of one another will have the appropriate line as a solution set. For example, let $(a,b,c) = (1,-1,1)$ in the first equation, and $(a,b,c) = (2,1,-4)$ in the second equation to obtain the system given here.

5. (a) $u = (2,2,1)$, since we know that the normal vector to the plane $ax + by + cz = d$ is (a,b,c).

(b) $v = (1,1,2)$.

(c) $\cos\theta = \frac{u \cdot v}{||u||\,||v||} = \frac{2}{\sqrt{6}}$. In MATLAB, type `acos(2/sqrt(6))*180/pi` to obtain $\theta = 35.2644°$.

9. Answer: $(x,y,z) = (1,3,-1)$.

SOLUTION The commands to instruct MATLAB to graph this three dimensional system are:

```
[x,y] = meshgrid(-10:0.5:10);
z = (-11 - 3*x + 4*y)/2;
surf(x,y,z)
hold on
z = 7 - 2*x - 2*y;
surf(x,y,z)
hold on
z = (7 + x - y)/(-5);
surf(x,y,z)
```

It is hard to determine a solution for the system from this graph. The command `axis([xmin xmax ymin ymax zmin zmax])` can make the graph clearer by zooming in on a specific range of points, but a numerically accurate solution is difficult to obtain graphically in three dimensions. Obtain an accurate solution using the command `A\b` in MATLAB.

Section 2.3

1. The matrix is not in reduced echelon form.

2. The matrix is in reduced echelon form.

3. The matrix is not in reduced echelon form.

4. The 1^{st} and 3^{rd} columns of the matrix contain pivots. The solutions of the system are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4x_2 \\ x_2 \\ 5 \end{pmatrix}$$

5. The 1^{st}, 3^{rd}, and 5^{th} columns of the matrix contain pivots. Since the last row of the matrix translates to the linear equation $0 = 1$, the system is inconsistent, and there are no solutions.

6. The 1^{st} and 3^{rd} columns of the matrix contain pivots. The solutions of the system are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 6x_2 \\ x_2 \\ 9 \\ x_4 \end{pmatrix}$$

8. **Answer:** The solution to this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 - 1 \\ x_3 \end{pmatrix},$$

where x_3 is any real number.

SOLUTION Row reduce the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right).$$

Although $(1, 2, 2)$ is not a solution to this system, there is a solution for which $x_3 = 2$, namely $(0, 1, 2)$.

9. The augmented matrix for this system is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 5 \\ 2 & 3 & -1 & 2 \end{array}\right)$$

We can row reduce this matrix to reduced echelon form, obtaining

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The last row of the reduced system implies that $0 = 1$, so the system is inconsistent and has no solutions.

10. The augmented matrix for this system is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 7 \end{array}\right)$$

which can be row reduced to

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{9}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{13}{5} \end{array}\right).$$

The solution set is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} - \frac{1}{5}x_3 - \frac{3}{5}x_4 \\ \frac{13}{5} + \frac{3}{5}x_3 - \frac{1}{5}x_4 \\ x_3 \\ x_4 \end{pmatrix}.$$

11. Answer: (a) The system has infinitely many solutions.

(b) One variable can be assigned arbitrary values.

SOLUTION The row-reduced form of the matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

12. Answer: The system has no solutions.

SOLUTION The row-reduced form of the matrix is:

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

13. Answer: The system has a unique solution.

SOLUTION The row-reduced form of the matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{3}{4} \end{array}\right).$$

14. Answer: (a) The system has infinitely many solutions.

(b) Two variables can be assigned arbitrary values.

SOLUTION The row-reduced form of the matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

20. Answer: The row echelon form matrix is

```
A =

    1.0000    0.5000    0.5000
         0         0    1.0000
```

SOLUTION Enter the augmented matrix into MATLAB as A, then reduce to row echelon form by typing:

```
A(1,:) = A(1,:)/2
A(2,:) = A(2,:) - 4*A(1,:)
```

The linear system that this matrix represents is inconsistent, since the 2^{nd} row of the reduced matrix represents the equation $0 = 1$.

21. The row-reduced matrix is:

```
A =

    1.0000   -1.3333         0    0.6667
         0    1.0000    1.5000    0.5000
         0         0    1.0000   -0.1429
```

This matrix represents a consistent linear system.

Section 2.4

1. The reduced echelon form of the matrix is:

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of A is two, since the reduced echelon matrix has two nonzero rows.

2. The reduced echelon form of the matrix is:

$$B = \begin{pmatrix} 1 & 0 & \frac{10}{3} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of B is two, since the reduced echelon matrix has two nonzero rows.

3. Answer: Four parameters are needed to specify all solutions.

SOLUTION According to Theorem **??**, $n - \ell$ parameters are needed to parameterize the set of all solutions of a linear system, where n is the number of unknowns, and ℓ is the rank of the reduced echelon matrix. In this case, $n = 7$ and $\ell = 3$.

4. Answer: Seven parameters are needed to specify all solutions.

SOLUTION According to Theorem **??**, $n - \ell$ parameters are needed to parameterize the set of all solutions of a linear system, where n is the number of unknowns, and ℓ is the rank of the reduced echelon matrix. In this case, $n = 12$ and $\ell = 5$.

5. Answer: Matrix A is consistent and requires 3 parameters to enumerate all solutions.

SOLUTION

```
rref(A) =
    1.0000         0    1.9474   -1.4211    2.2632    0.5789
         0    1.0000   -0.8947    0.8421   -0.5263   -0.1579
         0         0         0         0         0         0
         0         0         0         0         0         0
```

6. Answer: Matrix B is consistent and requires 1 parameter.

SOLUTION

```
rref(B) =
    1.0000    2.0000         0         0    1.0556
         0         0    1.0000         0   -0.2222
         0         0         0    1.0000   -0.1111
```

7. Answer: Matrix C is inconsistent.

SOLUTION

```
rref(C) =
     1     0    -5     0
     0     1     3     0
     0     0     0     1
```

10. The rank of the matrix is 3.

```
rref(A) =

    1.0000         0         0   -0.1429
         0    1.0000         0    1.2857
         0         0    1.0000         0
```