

Solutions, Assignment #3

Section 3.1

2.

$$By = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 + 20 - 2 \\ 2 + 10 - 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \end{pmatrix}$$

4. The product Ax cannot be computed.

5. $Ax = (13)$.

8. **Answer:** The equation $Cz = b$ is valid for $z = (\frac{2}{3}, \frac{1}{3})^t$.

SOLUTION Let $z = (z_1, z_2)^t$. Then $Cz = b$ implies

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which can be multiplied out, yielding the linear system

$$\begin{aligned} z_1 + z_2 &= 1 \\ 2z_1 - z_2 &= 1. \end{aligned}$$

This system can be solved by substitution to obtain $z_1 = \frac{2}{3}$ and $z_2 = \frac{1}{3}$.

11. **Answer:** The equations are valid when

$$A = \begin{pmatrix} 3 & 1 \\ -5 & 4 \end{pmatrix}.$$

SOLUTION Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

So

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

These matrix equations are equivalent to the linear equations

$$\begin{aligned} a_{11} &= 3 \\ a_{21} &= -5 \\ a_{12} &= 1 \\ a_{22} &= 4. \end{aligned}$$

12. Answer: The equations are valid when

$$A = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}.$$

SOLUTION Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

These matrix equations yield the linear system

$$\begin{aligned} a_{11} + a_{12} &= 2 \\ a_{21} + a_{22} &= -1 \\ a_{11} - a_{12} &= 4 \\ a_{21} - a_{22} &= 3, \end{aligned}$$

which can be written as an augmented matrix and row-reduced to yield the values a_{ij} :

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 & 3 \end{array} \right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right).$$

13. Answer: There is no 2×2 upper triangular matrix A that satisfies equation (3.1.6), but any symmetric matrix A of the form

$$A = \begin{pmatrix} 1 & 2 \\ 2 & a_{22} \end{pmatrix},$$

where a_{22} is a real number, satisfies (??).

SOLUTION Let A be the upper triangular matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

The resulting matrix equation

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

yields the linear equations

$$\begin{aligned} a_{11} &= 1 \\ 0 &= 2. \end{aligned}$$

The second equation is inconsistent, so there is no solution.

Then let A be the symmetric matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Write the matrix equation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

from which we obtain the consistent linear system

$$\begin{aligned} a_{11} &= 1 \\ a_{12} &= 2. \end{aligned}$$

Section 3.2

3. Answer: If $x = (x_1, 3x_1)^t$, where x_1 is any real scalar, then $Cx = 0$.

SOLUTION Solve $Cx = 0$ by row reducing C to find that $Cx = 0$ when $x_1 - \frac{1}{3}x_2 = 0$.

4. Answer:

$$R_{30^\circ} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

SOLUTION The 2×2 matrix that rotates the plane counterclockwise through an angle θ is

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

5. Answer:

$$R_{(-45^\circ)} = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

6. Answer:

$$2R_{(-90^\circ)} = \begin{pmatrix} 2 \cos(-90^\circ) & -2 \sin(-90^\circ) \\ 2 \sin(-90^\circ) & 2 \cos(-90^\circ) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

8. The map L_A that reflects vectors across the y -axis is $(x, y) \rightarrow (-x, y)$. The matrix is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

9. The map L_A that reflects vectors across the line $y = x$ is $(x, y) \rightarrow (y, x)$. The matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

18. Answer: Matrix A rotates the plane by $\theta \approx 1.1071$ counterclockwise and dilates it by a factor of $c = \sqrt{5} \approx 2.2361$.

SOLUTION Matrix A is a special case of Exercise ?? with $a = 1$ and $b = 2$. Thus, $c = \sqrt{a^2 + b^2} = \sqrt{5}$ and $\theta = \cos^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{1^2 + 2^2}} \right) \approx 1.1071$.

22. B rotates the plane 45° counterclockwise and reduces it by a factor of $\sqrt{2}$.

Section 3.3

3. The equation

$$(-2, -1) = \alpha(1, 2) + \beta(1, -3) = (\alpha + \beta, 2\alpha - 3\beta)$$

can be rewritten as the linear system

$$\begin{aligned} -2 &= \alpha + \beta \\ -1 &= 2\alpha - 3\beta \end{aligned}.$$

Solving this system yields $\alpha = -\frac{7}{5}$ and $\beta = -\frac{3}{5}$.

4. Answer: The vector $z = (2, 3, -1)$ cannot be written as

$$z = \alpha(2, 3, 0) + \beta(1, -1, 1).$$

SOLUTION Write the equation

$$(2, 3, -1) = \alpha(2, 3, 0) + \beta(1, -1, 1) = (2\alpha + \beta, 3\alpha - \beta, \beta)$$

as a linear system of two unknowns in three equations:

$$\begin{aligned} 2 &= 2\alpha + \beta \\ 3 &= 3\alpha - \beta \\ -1 &= \beta \end{aligned}.$$

Substituting $\beta = -1$ into the first and second equations yields

$$\begin{aligned} \frac{3}{2} &= \alpha \\ \frac{2}{3} &= \alpha. \end{aligned}$$

The system is inconsistent, so there is no solution to the desired equation.

5. Answer: The equation

$$\alpha(3, -2) + \beta(2, 3) + \gamma(1, 4) = (1, -2)$$

holds for any real numbers α, β, γ such that $\alpha = \frac{5}{13}\gamma + \frac{7}{13}$ and $\beta = -\frac{14}{13}\gamma - \frac{4}{13}$.

SOLUTION Write the equation as the linear system

$$\begin{aligned} 3\alpha + 2\beta + \gamma &= 1 \\ -2\alpha + 3\beta + 4\gamma &= -2. \end{aligned}$$

The augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -2 & 3 & 4 & -2 \end{array} \right)$$

row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{13} & \frac{7}{13} \\ 0 & 1 & \frac{14}{13} & -\frac{4}{13} \end{array} \right)$$

and the equation is valid for any values of α, β , and γ that satisfy this system.

6. Answer: The transformation $T(x, y, z) = (x + 2y - z, x - 4z)$ is linear.

SOLUTION Use the definition that a mapping is linear if it satisfies (??) and (??) to verify:

Let $X_1 = (x_1, y_1, z_1)$ and let $X_2 = (x_2, y_2, z_2)$. Verify (??) by computing $T(X_1 + X_2)$ and comparing it to $T(X_1) + T(X_2)$.

$$\begin{aligned} T(X_1 + X_2) &= T((x_1, y_1, z_1) + (x_2, y_2, z_2)) \\ &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= ((x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2), (x_1 + x_2) - 4(z_1 + z_2)) \\ &= (x_1 + x_2 + 2y_1 + 2y_2 - z_1 - z_2, x_1 + x_2 - 4z_1 - 4z_2) \\ &= (x_1 + 2y_1 - z_1, x_1 - 4z_1) + (x_2 + 2y_2 - z_2, x_2 - 4z_2). \end{aligned}$$

$$\begin{aligned} T(X_1) + T(X_2) &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= (x_1 + 2y_1 - z_1, x_1 - 4z_1) + (x_2 + 2y_2 - z_2, x_2 - 4z_2). \end{aligned}$$

Let $X = (x, y, z)$ and let c be any real scalar. Verify (??) by computing $cT(X)$ and comparing it to $T(cX)$.

$$\begin{aligned} cT(X) &= cT(x, y, z) \\ &= c(x + 2y - z, x - 4z) \\ &= (cx + 2cy - cz, cx - 4cz). \end{aligned}$$

$$\begin{aligned} T(cX) &= T(cx, cy, cz) \\ &= T(cX) \\ &= (cx + 2cy - cz, cx - 4cz). \end{aligned}$$

So $T(x, y, z) = (x + 2y - z, x - 4z)$ is a linear mapping.

8. The transformation $T(x, y) = (x + y, x - y - 1)$ is not linear because $T(0, 0) = (0, -1) \neq 0$, contradicting Theorem ??.

9. The transformation $T(x, y) = (1, x + y, 2y)$ is not linear because $T(0, 0) = (1, 0, 0) \neq 0$.

Section 3.4

2. **Answer:** Every solution can be written as a superposition of the vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

SOLUTION Write the matrix of the homogeneous system:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

This matrix cannot be row reduced further. Every solution has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_5 - x_4 - 2x_2 \\ x_2 \\ -x_5 + 2x_4 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

3. (a) **Answer:** All solutions to the homogeneous equation are of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$

SOLUTION Row reduce the matrix of the homogeneous system $Ax = 0$ to obtain:

$$\begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & -7 \end{pmatrix}.$$

So $x_1 = -11s$, $x_2 = 7s$ and $x_3 = s$.

(b) **Answer:** One possible solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

SOLUTION Assign a value to x_3 , then substitute into the two equations of the inhomogeneous system to obtain values for x_1 and x_2 .

(c) All solutions to (??) can be found by adding a single solution of the inhomogeneous system to all solutions of the homogeneous system, so:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -11 \\ 7 \\ 1 \end{pmatrix}.$$