Solutions, Assignment #6

Section 4.3

1. Answer: The characteristic polynomial of A is $p_A(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2$, and the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$.

SOLUTION Compute:

$$\begin{aligned}
p_A(\lambda) &= \det(A - \lambda I_3) \\
&= \begin{pmatrix} -9-\lambda & -2 & -10 \\ 3 & 2-\lambda & 3 \\ 8 & 2 & 9-\lambda \end{pmatrix} \\
&= (-9-\lambda)\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 9-\lambda \end{pmatrix} - 3\det\begin{pmatrix} -2 & -10 \\ 2 & 9-\lambda \end{pmatrix} + \\
&\quad 8\det\begin{pmatrix} -2 & -10 \\ 2-\lambda & 3 \end{pmatrix} \\
&= (-9-\lambda)(\lambda^2 - 11\lambda + 12) - 3(2 + 2\lambda) + 8(14 - 10\lambda) \\
&= -\lambda^3 + 2\lambda^2 + \lambda - 2 \\
&= (\lambda - 1)(\lambda + 1)(\lambda - 2).
\end{aligned}$$

The eigenvalues of A are the roots of the characteristic polynomial.

2. Answer: The characteristic polynomial of B is $p_B(\lambda) = \lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12$. The matrix B has single eigenvalues at $\lambda = 1$ and $\lambda = 3$ and a double eigenvalue at $\lambda = 2$.

SOLUTION Using Lemma 4.1.9, compute:

$$\begin{aligned}
p_B(\lambda) &= \det(B - \lambda I_3) \\
&= \det\begin{pmatrix} 2-\lambda & 1 & -5 & 2 \\ 1 & 2-\lambda & 13 & 2 \\ 0 & 0 & 3-\lambda & -1 \\ 0 & 0 & 1 & 1-\lambda \end{pmatrix} \\
&= \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \\
&= ((2-\lambda)^2 - 1)((3-\lambda)(1-\lambda) + 1) \\
&= (\lambda - 3)(\lambda - 1)(\lambda - 2)^2.
\end{aligned}$$

3. Answer: A basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 2$ is:

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

SOLUTION First, find all eigenvectors of A with eigenvalue $\lambda = 2$, that is, all vectors $v = (v_1, v_2, v_3)$ such that $(A - 2I_3)v = 0$. Solve the system

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

All solutions v to this system satisfy $v_1 = v_3 - v_2$. Thus:

$$v = \begin{pmatrix} v_3 - v_2 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, the vectors $(-1, 1, 0)^t$ and $(1, 0, 1)^t$ form a basis for this eigenspace.

5. (a) **Answer:** The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$, with corresponding eigenvectors $v_1 = (1, -1)^t$ and $v_2 = (1, -2)^t$, respectively.

SOLUTION The characteristic polynomial is $p_A(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$. Then, solve $Av = \lambda v$ for each eigenvalue to find the corresponding eigenvectors.

(b) Two linearly independent vectors in $\mathbb{R}^2$ form a basis for $\mathbb{R}^2$. Note that $v_1 \neq \alpha v_2$ for any scalar α. Therefore, v_1 and v_2 form a basis for $\mathbb{R}^2$.

(c) **Answer:** The coordinates of (x_1, x_2) in the basis $\{v_1, v_2\}$ are $(2x_1 + x_2, -x_1 - x_2)$.

SOLUTION Find α_1 and α_2 such that $\alpha_1 v_1 + \alpha_2 v_2 = (x_1, x_2)^t$. That is, solve:

$$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to obtain $\alpha_1 = 2x_1 + x_2$ and $\alpha_2 = -x_1 - x_2$.

6. **Answer:** The characteristic polynomial of A is $p_A(\lambda) = -(\lambda^3 + 5\lambda^2 + 6\lambda) = -\lambda(\lambda + 2)(\lambda + 3)$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -2$, and $\lambda_3 = -3$, with eigenvectors $v_1 = (0, 1, -1)^t$, $v_2 = (2, -1, 0)^t$, and $v_3 = (1, 0, -1)^t$, respectively.

SOLUTION The eigenvalues are the roots of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_3)$. The eigenvectors are vectors v such that $Av = \lambda v$, where λ is an eigenvalue of A. Find them by solving the system $(A - \lambda I_3)v = 0$.

7. We are given $A^2 + A + I_n = 0$. Therefore, $I_n = -A^2 - A = A(-A - I_n)$. Thus, $A^{-1} = -A - I_n$ exists.

11. (a) By calculation in MATLAB using the `eig`, `trace`, and `poly` commands, the eigenvalues of A are

$$\lambda = -0.5861 \pm 20.2517, \quad \lambda = -12.9416, \quad \lambda = -9.1033, \quad \text{and} \quad \lambda = 5.2171.$$

The trace of A is -18. The characteristic polynomial of A is

$$p_A = \lambda^5 + 18\lambda^4 + 433\lambda^3 + 6296\lambda^2 + 429\lambda - 252292.$$

Note that in order to obtain an accurate value for the characteristic polynomial, it may be necessary to use the `format` command.

(b) Theorem 4.2.7 states that the eigenvalues of A^{-1} are the inverses of the eigenvalues of A. In MATLAB , compute

```
eig(inv(A)) =
  -0.1098
  -0.0773
  -0.0014 + 0.0493i
  -0.0014 - 0.0493i
   0.1917
```

Then, compute the inverse of each eigenvalue of A to find that if λ is an eigenvalue of A, then λ^{-1} is indeed an eigenvalue of A^{-1}.

Section 4.4

1. $\lambda_1 = 2, \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 3, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. $\lambda_1 = -1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}; \quad \lambda_2 = 5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. $\lambda_1 = \lambda_2 = 1, \begin{pmatrix} 1 \\ - \end{pmatrix} 1$

9. $\lambda_1 = 2 + i, \begin{pmatrix} 1 \\ -i \end{pmatrix}; \quad \lambda_2 = 2 - i, \begin{pmatrix} 1 \\ i \end{pmatrix}$

10. $\lambda_1 = -1, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \quad \lambda_2 = 1, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \quad \lambda_3 = 2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

12. $\lambda_1 = 1, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_2 = \lambda_3 = -1, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

13. $\lambda_1 = \lambda_2 = \lambda_3 = 1, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

16 $\lambda_1 = 2, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_2 = 2+3i, \begin{pmatrix} -5+3i \\ 3+3i \\ 2 \end{pmatrix}; \quad \lambda_3 = 2-3i, \begin{pmatrix} -5-3i \\ 3-3i \\ 2 \end{pmatrix}$

17. $\lambda_1 = \lambda_2 = \lambda_3 = 1, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

21. $\lambda_1 = \lambda_2 = 2, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}; \quad \lambda_3 = 4, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_4 = 6, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$