Solutions, Assignment #7

Section 5.1

3. The set V_3 is a subspace of R^3 since the solution set to any equation $Ax = 0$ is a space. This is demonstrated by the principle of superposition introduced in Section **??**. Also, $V_3 = V_1$.

We can show that $V_3 = V_1$ by row reducing to find the solutions to $Ax = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

So all vectors in V_3 are of the form $x = s(-\frac{1}{2}, \frac{1}{2}, 1)$, where $s \in \mathbb{R}$. The vector x is an element of V_1 for each s.

5. Answer: W is not a subspace of V.

SOLUTION The subset W is closed neither under addition nor under scalar multiplication. For example, let $w_1 = (1, 4, 2)$ and $w_2 = (1, -1, 3)$ be elements of W. Then,

$$w_1 + w_2 = (1, 4, 2) + (1, -1, 3) = (2, 3, 5)$$

which is not an element of W.

6. Answer: W is not a subspace of V.

SOLUTION The subset W is closed neither under addition nor under scalar multiplication. For example, let $w_1 = (3, -2)$ and $w_2 = (0, 1)$ be elements of W. Then,

$$w_1 + w_2 = (3, -2) + (0, 1) = (3, -3).$$

The sum of the elements $3 - 3 = 0 \neq 1$.

7. W is a subspace of V, since W is closed under addition and scalar multiplication.

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9. W is a subspace of V, since W is closed under addition and scalar multiplication.

10. Answer: W is not a subspace of V.

SOLUTION The subset W is closed neither under addition nor under scalar multiplication. For example, let $w_1(t) = t$ and let $w_2(t) = t^2$.

$$(w_1 + w_2)(1) = w_1(1) + w_2(1) = 1 + 1 = 2,$$

so $(w_1 + w_2)(t)$ is not an element of W.

11. Answer: The set S is not a subspace.

SOLUTION The set S is closed under addition but not under scalar multiplication. To demonstrate, let $x = (1, 4, 2)$ be an element of S. Then

$$-2x = -2(1, 4, 2) = (-2, -8, -4)$$

which is not an element of S.

13. The set S is a subspace, since it is closed under addition and scalar multiplication.

Section 5.2

2. Answer: The subspace of solutions can be spanned by the vectors $(1, 1, 0)^t$ and $(-3, 0, 1)^t$.

SOLUTION All solutions to $x - y + 3z = 0$ can be written in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

5. Answer: The subspace of solutions is spanned by the vectors

$$(-2, 1, 0, 0, 0)^t \quad \text{and} \quad (-1, 0, -4, 1, 0)^t.$$

SOLUTION Let $x = (x_1, \ldots, x_5)$ be a solution to $Ax = 0$. All solutions to this equation have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}.$$

7. Answer: The subspace of solutions to $Ax = 0$ is spanned by the vector $(-2, -1, 1)^t$.

SOLUTION Let $x = (x_1, x_2, x_3)$ be a solution to $Ax = 0$. All solutions to this equation have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

9. Answer: The matrix A whose subspace of solutions in $\mathbb{R}^4$ is the span of v_1 and v_2 is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

SOLUTION Note that all vectors x in the spanning set of v_1 and v_2 are of the form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ b \\ -b \end{pmatrix}.$$

Therefore, $x_1 = -x_2$ and $x_3 = -x_4$. So,

$$\begin{array}{ccccccccc} x_1 & + & x_2 & & & & & = & 0 \\ & & & & x_3 & + & x_4 & = & 0. \end{array}$$

The matrix of this system is A.

11. Answer: The vector $(2, 20, 0)^t$ is in the span of w_1 and w_2. Specifically, $v = -4w_1 + 6w_2$.

SOLUTION Note that, for some real numbers a and b,

$$(2, 20, 0)^t = aw_1 + bw_2 = a(1, 1, 3)^t + b(1, 4, 2)^t$$

if v is in the span of w_1 and w_2. This corresponds to the linear system

$$\begin{array}{ccccc} a & + & b & = 2 \\ a & + & 4b & = 20 \\ 3a & + & 2b & = 0 \end{array}$$

To find a and b, row reduce the augmented matrix of the system:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 4 & 20 \\ 3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array}\right).$$

The system is consistent; $a = -4$ and $b = 6$.

12. Answer: The function $y(t) = 1 - t^2$ is an element of W and the set $\{y(t), x_2(t)\}$ is a spanning set for W.

SOLUTION The space W equals $\text{span}\{x_1(t), x_2(t)\}$ where $x_1(t) = 1$ and $x_2(t) = t^2$. To show that $y(t)$ is an element of W, let $a = 1$ and $b = -1$, and compute

$$ax_1(t) + bx_2(t) = x_1(t) - x_2(t) = 1 - t^2 = y(t).$$

To show that $\{y(t), x_2(t)\}$ is a spanning set for W, rewrite every linear combination of $x_1(t)$ and $x_2(t)$ in terms of $y(t)$ and $x_2(t)$, as follows:

$$ax_1(t) + bx_2(t) = a + bt^2 = a(1 - t^2) + (a + b)t^2 = ay(t) + (a + b)x_2(t).$$

13. The function $y(t) = t^4$ is not in W.

14. The function $y(t) = \sin(t)$ is not in W.

15. Answer: The function $y(t) = 0.5t^2$ is an element of W, but the set $\{y(t), x_2(t)\}$ does not span W.

SOLUTION When $a = 0$ and $b = 0.5$,

$$ax_1(t) + bx_2(t) = 0.5x_2(t) = 0.5t^2 = y(t).$$

In this case, there exist functions in W that are not in $\text{span}\{y(t), x_2(t)\}$. For example, the function $x_1(t) = 1$ cannot be written as a linear combination of $x_2(t)$ and $y(t)$.

Section 5.3

1. Type `null(A)` in MATLAB to find that the set of solutions to $Ax = 0$ is spanned by the vectors

$$\begin{pmatrix} 0.3225 \\ 0.8931 \\ -0.0992 \\ 0.2977 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -0.1961 \\ 0.5883 \\ 0.7845 \end{pmatrix}.$$

2. Answer: The solution to $Ax = 0$ is the vector $(0, 0)$.

SOLUTION This can be demonstrated by typing `null(A)` in MATLAB , which yields

```
ans =

   Empty matrix: 2-by-0
```

5. Answer: The solution set of $Bx = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + \frac{3}{4}x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{3}{4} \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

SOLUTION Row reduce B:

$$\begin{pmatrix} -4 & 0 & 4 & 3 \\ -4 & 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & -\frac{3}{4} \\ 0 & 1 & 3 & -2 \end{pmatrix}.$$

The solution obtained by row reduction is not the same as the one obtained using `null`, but the solution vectors are linear combinations of the MATLAB solution vectors, so the answers are equivalent. By row reducing the matrix `[null(B) x]`, where $x = (-1, -3, 1, 0)$, we find that

$$\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} = -3.1009 \begin{pmatrix} 0.3225 \\ 0.8931 \\ -0.0992 \\ 0.2977 \end{pmatrix} + 1.1767 \begin{pmatrix} 0 \\ -0.1961 \\ 0.5883 \\ 0.7845 \end{pmatrix}.$$

By row reducing the matrix `[null(B) y]` where $y = (\frac{3}{4}, 2, 0, 1)$ we find that:

$$\begin{pmatrix} \frac{3}{4} \\ 2 \\ 0 \\ 1 \end{pmatrix} = 2.3257 \begin{pmatrix} 0.3225 \\ 0.8931 \\ -0.0992 \\ 0.2977 \end{pmatrix} + 0.3922 \begin{pmatrix} 0 \\ -0.1961 \\ 0.5883 \\ 0.7845 \end{pmatrix}.$$