Solutions, Assignment #8

Section 5.4

1. To show that the set of vectors $\{w_1, w_2\}$ is linearly dependent, show that there exist nonzero a and b such that $aw_1 + bw_2 = 0$. For the set $\{w, 0\}$, if $a = 0$ and $b = 1$, then $0w + 1(0) = 0$, so the set is linearly dependent. For the set $\{w, -w\}$, if $a = 1$ and $b = 1$, then $w - w = 0$, so the set is linearly dependent.

2. Answer: The set is linearly independent if $b \neq -\frac{1}{3}$.

SOLUTION Note that a set of two vectors is linearly dependent if one is a multiple of the other. So this set is dependent for any values of b at which

$$(3, -1) = \alpha(1, b).$$

When equality holds $\alpha = 3$. Therefore, $b = -\frac{1}{3}$.

3. Answer: The set is linearly dependent.

SOLUTION Let A be the matrix whose columns are u_1, u_2, and u_3. The set $\{u_1, u_2, u_3\}$ is linearly dependent if there exists a nonzero vector $r = (r_1, r_2, r_3)$ such that $r_1u_1 + r_2u_2 + r_3u_3 = 0$, that is, if the homogeneous system $Ar = 0$ has a nonzero solution. Row reduce:

$$\begin{pmatrix} 1 & 2 & 10 \\ -1 & 1 & 2 \\ 1 & -2 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $Ar = 0$ when $r = r_3(-2, -4, 1)$. The value of r is nonzero for $r_3 \neq 0$, so the set is indeed linearly dependent. As an example, let $r_3 = 1$. Then,

$$-4u_1 - 2u_2 + u_3 = -2(1, -1, 1) - 4(2, 1, -2) + (10, 2, -6) = (0, 0, 0) = 0.$$

4. Answer: The vectors $(1, b, 2b)$ and $(2, 1, 4)$ are linearly independent for any value of b.

SOLUTION Two vectors are linearly independent unless one is a multiple of the other; in this case, unless

$$(1, b, 2b) = \alpha(2, 1, 4).$$

Equality holds if $2\alpha = 1$, $\alpha = b$, and $4\alpha = 2b$. Therefore, $\alpha = \frac{1}{2}$, $b = \frac{1}{2}$ and $b = 1$, which is inconsistent, so the vectors are linearly independent.

5. Answer: The polynomials $p_1(t) = 2 + t$, $p_2(t) = 1 + t^2$, and $p_3(t) = t - t^2$ are linearly independent in $\mathcal{C}^1$.

SOLUTION We can determine this by noting that the polynomials are linearly dependent if there exists a nonzero vector $r = (r_1, r_2, r_3)$ such that $r_1p_1 + r_2p_2 + r_3p_3 = 0$. It is convenient to represent each polynomial as a vector $(a, b, c) = p(t) = a + bt + ct^2$. Thus, $p_1(t) = (2, 1, 0)$, $p_2(t) = (1, 0, 1)$, and $p_3(t) = (0, 1, -1)$. Solve the homogeneous system $Ar = 0$, where A is the matrix whose columns are p_1, p_2, and p_3, by row reduction.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, there are no nonzero values of r for which $r_1p_1 + r_2p_2 + r_3p_3 = 0$, and the polynomials are linearly independent.

6. The three functions are linearly dependent vectors in $\mathcal{C}^1$ since there exists a nonzero vector $r = (r_1, r_2, r_3)$ such that $r_1f_1(t) + r_2f_2(t) + r_3f_3(t) = 0$. We can find this vector r using trigonometric identities:

$$f_3(t) = \cos\left(t + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos t + \sin\left(\frac{\pi}{3}\right)\sin t = \frac{1}{2}\cos t - \frac{\sqrt{3}}{2}\sin t = \frac{1}{2}f_1(t) - \frac{\sqrt{3}}{2}f_2(t).$$

That is, $\frac{1}{2}f_1(t) + \frac{\sqrt{3}}{2}f_2(t) - f_3(t) = 0$.

8. Answer: The set $\{v_1, v_2, v_3\}$ is linearly dependent.

SOLUTION The set is linearly dependent if there exist scalars r_1, r_2, and r_3 such that $r_1v_1 + r_2v_2 + r_3v_3 = 0$. Create a matrix `A` whose columns are v_1, v_2 and v_3. Then row reduce `A` to solve the homogeneous system $Ar = 0$. Specifically, row reducing the matrix `A = [v1 v2 v3]` yields

```
ans =
     1     0     5
     0     1     2
     0     0     0
     0     0     0
```

So $-5v_1 - 2v_2 + v_3 = 0$.

10. Answer: The set $\{x_1, x_2, x_3\}$ is linearly independent.

SOLUTION The matrix `A` associated to the set $\{x_1, x_2, x_3\}$ row reduces to

```
ans =
     1     0     0
     0     1     0
     0     0     1
     0     0     0
     0     0     0
```

In this case, there are no nonzero solutions to $r_1x_1 + r_2x_2 + r_3x_3 = 0$.

Section 5.4A

1. Linearly dependent; $\mathbf{v}_3 = 2\mathbf{v}_2 + 3\mathbf{v}_2$.

2. Linearly independent.

3. Linearly dependent; $\mathbf{v}_4 = \mathbf{v}_2 + 2\mathbf{v}_2 - 3\mathbf{v}_3$.

6. Linearly independent.

7. Linearly dependent; $\mathbf{v}_3 = 2\mathbf{v}_2$.

9. $b = -1$.

14. $W(\sin\, ax,\ \cos\, ax = -a \neq 0)$; linearly indepdentent.

15. $W(x, x^2, x^3) = 2x^3 \neq 0$; linearly independent.

Section 5.5

1. By Theorem **??**, $\mathcal{U}$ is a basis for $\mathbb{R}^3$ if the vectors of $\mathcal{U}$ are linearly independent and span $\mathbb{R}^3$. By Lemma **??**, the dimension of $\mathcal{U}$ is equal to the rank of the matrix whose rows are u_1, u_2, and u_3. Row reduce this matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\dim(\mathcal{U}) = 3 = \dim(\mathbb{R}^3)$, and we need now only show that u_1, u_2, and u_3 are linearly independent, which we can do by row reducing the matrix whose columns are the vectors of $\mathcal{U}$ as follows:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, there is no nonzero solution to the equation $\mathcal{U}r = 0$, so the vectors of $\mathcal{U}$ are linearly independent and $\mathcal{U}$ is a basis for $\mathbb{R}^3$.

2. Answer: The dimension of S is 2, and vectors v_1 and v_2 form a basis for S.

SOLUTION Row reduce the matrix A whose rows are v_1, v_2, and v_3. By Lemma **??**, the number of nonzero rows in the reduced matrix is the dimension of S and these rows form a basis for S. So:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & -1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Answer: The vectors $(1, 1, 1, 0)$ and $(-2, -2, 0, 1)$ form a basis for the nullspace of A; therefore the dimension of the nullspace is 2.

SOLUTION Find the set of solutions to $Ax = 0$ by solving

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & -1 & 0 & 0 \\ 4 & -5 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.$$

To solve, row reduce A, obtaining

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the set of solutions to $Ax = 0$ can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - 2x_4 \\ x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

4. The set V is a vector space because the operations of addition and scalar multiplication satisfy the eight properties of vector spaces described in Table **??**. For 2×2 matrices, matrix addition is defined for two matrices such that:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

and scalar multiplication is defined for a matrix and a scalar such that:

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}.$$

So, using these definitions, addition is commutative and associative, and the additive identity is the 2×2 matrix of zeroes. If

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \text{then} \quad W^{-1} = \begin{pmatrix} -w_{11} & -w_{12} \\ -w_{21} & -w_{22} \end{pmatrix}.$$

Scalar multiplication is associative. There is a multiplicative identity, I_2, and scalar multiplication is distributive both for scalars and for matrices. So V is a vector space. One basis for V is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of $m \times n$ matrices is also a vector space, since it also satisfies the eight properties of vector spaces. In this case, the additive identity is the $m \times n$ zero matrix, and the multiplicative identity is I_n. The dimension of the set is mn, since one basis consists of the mn matrices with $a_{ij} = 1$ and all other entries 0, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

5. The set P_n is a subspace if it is closed under addition and scalar multiplication. Let $x(t) = a_0 + a_1 t + \cdots + a_n t^n$, $y(t) = b_0 + b_1 t + \cdots + b_n t^n$ and $s \in \mathbb{R}$. Then

$$x(t) + y(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \in P_n.$$
$$cx(t) = c(a_0 + a_1 t + \cdots + a_n t^n) = ca_0 + ca_1 t + \cdots + ca_n t^n \in P_n.$$

The dimension of P_2 is 3, since $x_1 = 1$, $x_2 = t$, and $x_3 = t^2$ form a basis for P_2. The dimension of P_n is $n + 1$.

Section 5.6

1. Answer: The span of v_1 and v_2 is a plane with normal vector $N = n_3(-\frac{3}{2}, 1, 1)$, where n_3 is a nonzero scalar.

SOLUTION If v_1 and v_2 are linearly independent, then they span a plane in $\mathbb{R}^3$. If they are linearly dependent, that is, if $v_1 = \alpha v_2$ for some scalar α, then they span a line in $\mathbb{R}^3$. In this case, there is no scalar α such that

$(2,1,2) = \alpha(0,-1,1)$, so the span of v_1 and v_2 has dimension two. The vector $N = (n_1, n_2, n_3)$ is found by observing that:

$$\begin{array}{rcrcrcl} 2n_1 & + & n_2 & + & 2n_3 & = & 0 \\ & & -n_2 & + & n_3 & = & 0 \end{array}$$

which is a linear system in two equations. Solve for N by row reducing the corresponding matrix:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{pmatrix}.$$

2. Answer: The subspace spanned by v_1 and v_2 is a line, since, if $\alpha = -\frac{1}{2}$, then $(2,1,-1) = \alpha(-4,-2,2)$.

3. Answer: The span of v_1 and v_2 is a plane with normal vector $N = n_3(0,0,1)$, where n_3 is a nonzero scalar.

SOLUTION There is no scalar α such that $(0,1,0) = \alpha(4,1,0)$. Let $N = (n_1, n_2, n_3)$ be the vector perpendicular to the plane. Then:

$$\begin{array}{rcrcl} & & n_2 & = & 0 \\ 4n_1 & + & n_2 & = & 0 \end{array}$$

Solve for N by substitution to find that $n_1 = n_2 = 0$, and n_3 can be any nonzero real scalar.

9. The span has dimension 3 for $\lambda \neq 2$, and the set $\{w_1, w_2, w_3\}$ is a basis for $\mathbb{R}^3$.

SOLUTION Find the dimension of the span by creating a matrix with rows w_1, w_2, w_3, and w_4, then row reducing:

$$\begin{pmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ 3 & -2 & \lambda \\ -5 & 6 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \lambda - 2 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (1)$$

If $\lambda = 2$, then the dimension of the span will be 2 and if $\lambda \neq 2$, then the dimension of the span will be 3. For example, let $\lambda = -1$.

Verify by row reduction that the set $\{w_1, w_2, w_3\}$ is a basis for $\mathbb{R}^3$ and that the set $\{w_1, w_2, w_4\}$ is not a basis for R^3.

(b) If $\lambda = 2$, then the dimension of span$\{w_1, w_2, w_3, w_4\}$ is 2, as shown by equation (1).

10. Here is a sample MATLAB output for this problem. Type:

```
x1 = rand(5,1);
x2 = rand(5,1);
x3 = rand(5,1);
x4 = rand(5,1);
x5 = rand(5,1);
```

A summary of the results is:

```
x1 =          x2 =          x3 =          x4 =          x5 =
    0.9501        0.7621        0.6154        0.4057        0.0579
    0.2311        0.4565        0.7919        0.9355        0.3529
    0.6068        0.0185        0.9218        0.9169        0.8132
    0.4860        0.8214        0.7382        0.4103        0.0099
    0.8913        0.4447        0.1763        0.8936        0.1389
```

The command `A = [x1 x2 x3 x4 x5]` creates the matrix with columns $x_1, ..., x_5$. Type `rref(A)` to verify that the vectors are linearly independent. The following steps display the vector $b = (2, 1, 3, -2, 4)$ as a linear combination of $x_1, ..., x_5$. Type:

```
b = [2;1;3;-2;4];
C = [A b];
rref(C)
```

which yields:

```
ans =
    1.0000         0         0         0         0   28.7614
         0    1.0000         0         0         0  -96.6468
         0         0    1.0000         0         0   70.9112
         0         0         0    1.0000         0   30.0838
         0         0         0         0    1.0000 -129.8826
```

So, in this example, $b = 28.7614x_1 - 96.6468x_2 + 70.9112x_3 + 30.0838x_4 - 129.8826x_5$.