Solutions, Assignment #9

Section 6.1

1. Compute A, the matrix of L, using Equation (**??**):

$$A = (w_1^t|w_2^t|w_3^t)(v_1^t|v_2^t|v_3^t)^{-1} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -7 & -11 & 3 \\ -4 & -7 & 2 \end{pmatrix}.$$

7. Let X and Y be elements of $\mathcal{M}(n)$. Then,

$$L(X+Y) = A(X+Y)-(X+Y)A = (AX-XA)+(AY-YA) = L(X)+L(Y).$$

For any real scalar c,

$$L(cX) = A(cX) - (cX)A = c(AX) - c(XA) = c(AX - XA) = cL(X).$$

Therefore, L is a linear mapping.

The null space of L consists of all matrices X such that $AX - XA = 0$, or $AX = XA$. By definition, these are the matrices such that X commutes with A. Let X and Y be elements of the null space of L. Then show that $X + Y$ is in the null space by calculating:

$$L(X+Y) = A(X+Y)-(X+Y)A = AX+AY-XA-YA = AX+AY-AX-AY = 0.$$

Show that, for any real scalar c, cX is in the null space by calculating:

$$L(cX) = A(cX) - (cX)A = cAX - cXA = cAX - cAX = 0.$$

Therefore, the null space of L is a subspace consisting of all matrices that commute with A.

Section 6.2

1. **Answer:** The possible choices for the scalars α_j are $\alpha = (\alpha_1, \alpha_2, \alpha_3) = \alpha_3(-1, -1, 1)$ and the possible choices for the scalars β_j are $\beta = (\beta_1, \beta_2, \beta_3) = \beta_3(-\frac{7}{5}, -\frac{9}{5}, 1)$.

SOLUTION Find A^t and solve by row reduction the equation $A^t\alpha = 0$. To find the scalars β_j, solve $A\beta = 0$. These equations yield

$$-r_1 - r_2 + r_3 = 0 \quad \text{and} \quad -7c_1 - 9c_2 + 5c_3 = 0.$$

2. The largest row rank that a 5×3 matrix can have is 3, since, by Theorem **??** the row rank is equal to the column rank, and the matrix has 3 columns.

3. (a) **Answer:** The vectors $(1, 0, 1, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 0, 1)$ form a basis for the row space of A, and the row rank of A is 3.

SOLUTION Row reduce A:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & 2 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) **Answer:** The column rank of A is 3, and the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ form a basis for the column space of A.

SOLUTION Row reduce A^t:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) **Answer:** The vector $(-1, 1, 1, 0)$ is a basis for the null space. Since one vector forms the basis, the nullity of A is 1.

SOLUTION Solve $Ax = 0$ by row reducing A, which we have already done.

(d) **Answer:** The null space is trivial and the nullity of A^t is 0.

SOLUTION Find a basis by solving $A^t x = 0$ by row reduction. The row reduced matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

implies $x = (0, 0, 0)$.

Section 6.3

1. Answer: $[v]_{\mathcal{W}} = (7, 4)$.

SOLUTION Find the scalars α_1 and α_2 such that $v = \alpha_1 w_1 + \alpha_2 w_2$. That is, solve the linear system

$$\begin{array}{rcrcr} \alpha_1 & - & 2\alpha_2 & = & -1 \\ 4\alpha_1 & + & \alpha_2 & = & 32 \end{array}$$

to obtain $(\alpha_1, \alpha_2) = (7, 4)$, the coordinates of v in the $\mathcal{W}$ basis.

3. (a) By Theorem **??**, the subset $\mathcal{V}$ is a basis for the vector space of 2×3 matrices if the vectors of $\mathcal{V}$ are linearly independent and span the vector space. Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}.$$

We show that B is in the span of $\mathcal{V}$ by noting that $B = b_{11}E_{11} + b_{12}E_{12} + b_{13}E_{13} + b_{21}E_{21} + b_{22}E_{22} + b_{23}E_{23}$. To show that the matrices E_{ij} are linearly independent, suppose $b_{11}E_{11} + b_{12}E_{12} + b_{13}E_{13} + b_{21}E_{21} + b_{22}E_{22} + b_{23}E_{23} = 0$. Then,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = 0,$$

so $b_{ij} = 0$. Therefore, V is a basis for the given vector space.

(b) **Answer:** $[A]_{\mathcal{V}} = (-1, 0, 2, 3, -2, 4)$.

SOLUTION Compute $A = -E_{11} + 2E_{13} + 3E_{21} - 2E_{22} + 4E_{23}$.

5. **Answer:** $[v]_W = (-2, 2, -1)$.

SOLUTION Use MATLAB to row reduce the augmented matrix $(w_1^t | w_2^t | w_3^t | v)$, obtaining:

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ans =
     1     0     0    -2
     0     1     0     2
     0     0     1    -1
```

7. **Answer:** The matrix $[L]_{\mathcal{W}}$ is diagonal in the basis:

$$\mathcal{W} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

SOLUTION Theorem **??** states that the matrix $[L]_{\mathcal{W}}$ is diagonal if $\mathcal{W}$ consists of eigenvectors of L corresponding to real eigenvalues. By computation, we find that $\lambda_1 = 2$ and $\lambda_2 = -1$ are the eigenvalues of L. We then find that $Lw_1 = 2w_1$ when $w_1 = (1, 2)^t$ and $Lw_2 = -w_2$ when $w_2 = (2, 3)^t$, so w_1 and w_2 are the eigenvectors of L.

Section 6.4

1. Answer:

$$C_{\mathcal{W}\mathcal{Z}} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

SOLUTION Substitute into Equation (**??**) as follows:

$$C_{\mathcal{W}\mathcal{Z}} = (w_1^t|w_2^t)^{-1}(z_1^t|z_2^t) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$