

MATH 3321(H)

Homework Assignment #4

Due date: Friday, February 17, 2012

NAME (Print): \_\_\_\_\_

Instructions:

- Print out this file and complete all the problems.
  - Write your solutions in the space provided.
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1. Given the differential equation  $y'' - \left(\frac{2}{x}\right)y' - \left(\frac{4}{x^2}\right)y = 0$ .

(a) Find values of  $r$  such that  $y = x^r$  is a solution of the equation. Determine a fundamental set of solutions and give the general solution of the equation.

(b) Find the solution of the equation satisfying the initial conditions  $y(1) = 2$ ,  $y'(1) = -1$ .

(c) Find the solution of the equation satisfying the initial conditions  $y(2) = y'(2) = 0$ .

2. Given the differential equation  $(x^2 + 2x - 1)y'' - 2(x + 1)y' + 2y = 0$ .

(a) Show that the equation has a linear polynomial and a quadratic polynomial as solutions.

b Give the general solution of the equation.

3. Show that  $y_1(x) = e^{3x}$ ,  $y_2(x) = e^{-x}$ ;  $I = (-\infty, \infty)$  are linearly independent on  $I$  and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

4. Show that  $y_1(x) = x^2$ ,  $y_2(x) = x^2 \ln x$ ;  $I = (0, \infty)$  are linearly independent on  $I$  and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

5. Let  $y = y_1(x)$  be a solution of (H):  $y'' + p(x)y' + q(x)y = 0$  where  $p$  and  $q$  are continuous function on an interval  $I$ . Assume that  $y_1(x) \neq 0$  on  $I$  and set

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(u) du}}{y_1^2(t)} dt. \quad (\text{ignore constants of integration})$$

Show that  $y_2$  is a solution of (H) and that  $y_1$  and  $y_2$  are linearly independent.

6. Given that  $y_1(x) = e^{x^2}$  is a solution of  $y'' - \frac{1}{x}y' - 4x^2y = 0$ , find the general solution of the equation.

7. Given that  $y_1(x) = e^x$  is a solution of  $y'' - \frac{2x-1}{x}y' + \frac{x-1}{x}y = 0$ , find the general solution of the equation.

8. Let  $y = y_1(x)$  and  $y = y_2(x)$  be solutions of equation (H) on an interval  $I$ . Let  $a \in I$  and suppose that

$$y_1(a) = \alpha, y_1'(a) = \beta \quad \text{and} \quad y_2(a) = \gamma, y_2'(a) = \delta.$$

Under what conditions on  $\alpha, \beta, \gamma, \delta$  will the functions  $y_1$  and  $y_2$  be linearly independent on  $I$ ?

9. Suppose that the functions  $y_1$  and  $y_2$  are linearly independent solutions of (H). Under what conditions will the functions  $y_3 = \alpha y_1 + \beta y_2$  and  $y_4 = \gamma y_1 + \delta y_2$  be linearly independent solutions of (H)?

10. Find the general solution of each of the following differential equations:

(a)  $y'' - 8y' + 16y = 0$

(b)  $y'' - 3y' - 10y = 0$

(c)  $y'' + 4y' + 20y = 0$

11. Given the differential equation  $y'' - (2a - 1)y' + a(a - 1)y = 0$ .

(a) Determine the values of  $a$  (if any) for which all solutions have limit 0 as  $x \rightarrow \infty$ .

(b) Determine the values of  $a$  (if any) for which all solutions are unbounded as  $x \rightarrow \infty$ .

(c) Determine the values of  $a$  (if any) for which some solutions have limit 0 and other solutions are unbounded as  $x \rightarrow \infty$ .

Exercises 12 - 14 are concerned with the differential equation (1):  $y'' + ay' + by = 0$  where  $a$  and  $b$  are constants.

12. Prove that if  $a$  and  $b$  are both positive, then all solutions have limit 0 as  $x \rightarrow \infty$ .

13. Prove that if  $a = 0$  and  $b > 0$ , then all solutions of the equation are bounded.

14. Prove that if  $a > 0$  and  $b = 0$ , and  $y = y(x)$  is a solution, then

$$\lim_{x \rightarrow \infty} y(x) = k \quad \text{for some constant } k.$$

Determine  $k$  for the solution that satisfies the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta.$$

Find the general solution of the Euler equations 15 - 17.

15.  $x^2 y'' - xy' - 8y = 0$ .

16.  $x^2y'' - 3xy' + 4y = 0$ .

17.  $x^2y'' - xy' + 5y = 0$ .