Assignment 6 Chapter 4.

Due date: **June 18**

Section 4.3. 1, 2, 3, 5, 6, 7, 11

Section 4.4. 1, 4, 5, 9, 10, 12, 13, 16, 17, 21

Problems to be turned in for grading.

1. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

   (a) Verify that the characteristic polynomial of A is $p_\lambda(A) = (\lambda - 1)(\lambda + 2)^2$.

   (b) Show that $(1, 1, 1)$ is an eigenvector of A corresponding to $\lambda = 1$.

   (c) Show that $(1, 1, 1)$ is orthogonal to every eigenvector of A corresponding to the eigenvalue $\lambda = -2$.

2. Decide whether the given statement is *true* or *false*. If the statement is false, give a counterexample; if the statement is true, give a proof.

   (a) If the eigenvalues of a 2×2 matrix are equal to 1, then the four entries of that matrix are each less than 500.

   (b) The trace of the product of two $n \times n$ matrices is the product of the traces.

3. Show that every real $n \times n$ matrix has a real eigenvalue when n is odd.

4. (Computer exercise) Compute the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 4 & -6 & 7 \\ 2 & 0 & 5 \\ -10 & 2 & 5 \end{pmatrix}$$

   Denote this polynomial by $p_A(\lambda) = -(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)$. Then compute the matrix

$$B = -(A^3 + p_2A^2 + p_1A + p_0I).$$

   What do you observe? In symbols $B = p_A(A)$. Compute the matrix B for examples of other square matrices A and determine whether or not your observation was an accident.

5. Prove that if λ is an eigenvalue of A with eigenvector $\mathbf{v}$, then for every positive integer k, λ^k is an eigenvalue of A^k with eigenvector $\mathbf{v}$.

6. Suppose that λ is a nonzero eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Prove that if A has an inverse, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding vector $\mathbf{v}$.

7. Find three 2×2 matrices A having eigenvalues $\lambda_1 = 4$, $\lambda_2 = 5$.

8. Choose $a,\ b,\ c$ so that the 3×3 matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$$

has eigenvalues $\lambda_1 = -3,\ \lambda_2 = 0,\ \lambda_3 = 3.$

9. An $n \times n$ matrix A is said to be *nilpotent* if $A^k = 0$ for some positive integer k. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

10. Given the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$. Show that if $a + b = c + d$, then $(1, 1)$ is an eigenvector of A. Find both eigenvalues of A.