Problem 1. (a) (*i*) $\text{dom}\,(f) = \{(x,y) | x^2 + y^2 < 9\}$ (*ii*) $\frac{1}{3}$

(b) (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 1 (*v*) No

Problem 2.

(a) $f_{xx} = y^4 e^{xy}$; $f_{yx} = f_{xy} = 3y^2 e^{xy} + xy^3 e^{xy} - \dfrac{1}{y^2}$

(c) $\dfrac{du}{dt} = 2x \cos t - 8y\, e^{2t} + 9z^2 = 2 \sin t \cos t - 8e^{4t} + 81t^2$ (all in terms of t)

(d) $\dfrac{\partial z}{\partial u} = \left(2e^{2x} \ln y\right)(2u) + \left(\dfrac{1}{y}\, e^{2x}\right)(-2)$; $\dfrac{\partial z}{\partial v} = \left(2e^{2x} \ln y\right)(-2) + \left(\dfrac{1}{y}\, e^{2x}\right)(2v)$

Problem 3.

(a) (*i*) $\nabla f = \left(x^2 y e^{x-1} + 2xye^{x-1} + 2y^2\right) \mathbf{i} + \left(x^2 e^{x-1} + 4xy\right) \mathbf{j}$

(*ii*) $\nabla f(1,-1) = -\mathbf{i} - 3\,\mathbf{j}$; direction of most rapid decrease: $-\nabla f(-1,1) = \mathbf{i} + 3\,\mathbf{j}$; most rapid decrease: $-||\nabla f|| = -\sqrt{10}$

(b) (*i*) $\nabla F = (2x + 4y)\,\mathbf{i} + (3z + 4x)\,\mathbf{j} + 3y\,\mathbf{k}$

(*ii*) $F'_{\mathbf{u}}(1,1,-5) = (6\,\mathbf{i} - 11\,\mathbf{j} + 3\,\mathbf{k}) \cdot \left(\frac{1}{2}\,\mathbf{i} + \frac{3}{4}\,\mathbf{j} - \frac{1}{4}\sqrt{3}\,\mathbf{k}\right) = \dfrac{-21 - 3\sqrt{3}}{4}$

(c) tangent plane: $2(x-3) + 6(y+1) - 3(z+2) = 0$

(d) tangent plane: $-(x-1) - 3(y+1) - (z-1) = 0$

normal line: $x = 1 - t,\ y = -1 - 3t,\ z = 1 - t$

Problem 4.

(a) No; $\dfrac{\partial P}{\partial y} = 6x^2 y + 3$, $\dfrac{\partial Q}{\partial x} = 6x^2 y + 3y$; $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$

(b) Yes, $\dfrac{\partial P}{\partial y} = 2x\, e^y = \dfrac{\partial Q}{\partial x}$; $f(x,y) = x^2 e^y + \dfrac{1}{2}\, e^{2x} - 3x + \dfrac{1}{2} \sin 2y + y + C$, C any constant.

Problem 5.

(a) $f_x = x^2 + 2y - 7$, $f_y = y + 2x - 1$

Solve the system of equations:

$$x^2 + 2y - 7 = 0$$

$$y + 2x - 1 = 0$$

Solutions: $(5,-9),\ (-1,3)$

(b) $f_{xx} = 2x, \quad f_{xy} = 2, \quad f_{yy} = 1, \quad D = AC - B^2$

point	A	B	C	D	result
$(5,-1)$	10	2	1	6	loc. min
$(-1,3)$	-2	2	1	-6	saddle

Problem 6.

(a) $\nabla f = (2x-2)\,\mathbf{i} + (2y-2)\,\mathbf{j} = \mathbf{0}$ at $(1,1) \in D$.

The boundary of D is given by: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le 2\pi.$

f on the boundary is given by:

$$f(\mathbf{r}(t)) = g(t) = (2\cos t)^2 + (2\sin t)^2 - 4\cos t - 4\sin t + 2 = 6 - 4\cos t - 4\sin t.$$

$$g'(t) = 4\sin t - 4\cos t; \quad g'(t) = 0 \implies \sin t = \cos t \implies t = \pi/4,\ t = 5\pi/4.$$

The corresponding points are: $\left(\sqrt{2},\sqrt{2}\right)$, $\left(-\sqrt{2},-\sqrt{2}\right)$ and the endpoint $g(0) = g(2\pi):\ (2,0)$

$f(1,1) = 0$ (abs. min.), $f\left(\sqrt{2},\sqrt{2}\right) = 6 - 4\sqrt{2} \cong 0.343$, $f\left(-\sqrt{2},-\sqrt{2}\right) = 6 + 4\sqrt{2} \cong 11.657$ (abs. max.), $f(2,0) = 2$

(b) $\nabla f = (2-2x)\,\mathbf{i} + (2-2y)\,\mathbf{j} = \mathbf{0}$ at $(1,1) \in D$; $f(1,1) = 4$.

The boundary of D consists of the three sides of the triangle:

$$C_1: \quad 0 \le x \le 9: \quad \mathbf{r}(t) = t\,\mathbf{i}, \quad 0 \le t \le 9,$$

$$C_2: \quad x + y = 9: \quad \mathbf{r}(t) = t\,\mathbf{i} + (9-t)\,\mathbf{j}, \quad 0 \le t \le 9,$$

$$C_3: \quad 0 \le y \le 9: \quad \mathbf{r}(t) = t\,\mathbf{j}, \quad 0 \le t \le 9.$$

On C_1: $f(\mathbf{r}(t)) = g(t) = 2 + 2t - t^2; \quad g'(t) = 2 - 2t; \quad g'(t) = 0 \implies t = 1$

$$g(0) = f(0,0) = 2, \quad g(1) = f(1,0) = 3, \quad g(9) = f(9,0) = -61$$

On C_2: $f(\mathbf{r}(t)) = g(t) = -2t^2 + 18t - 61; \quad g'(t) = 0 \implies t = \frac{9}{2}$

$$g(0) = f(0,9) = -61, \quad g(9/2) = f\left(\tfrac{9}{2},\tfrac{9}{2}\right) = -\frac{41}{2}, \quad g(9) = f(9,0) = -61$$

On C_3: $f(\mathbf{r}(t)) = g(t) = 2 + 2t - t^2; \quad g'(t) = 2 - 2t; \quad g'(t) = 0 \implies t = 1$

$$g(0) = f(0,0) = 2, \quad g(1) = f(0,1) = 3, \quad g(9) = f(0,9) = -61$$

The absolute max of f is: $f(1,1) = 4$; the absolute min is: $f(9,0) = f(0,9) = -61$.

Problem 7.

(a) Let the dimensions of the box be: length – x, width – y, height – z.

Maximize the volume: $V = xyz$ subject to the constraint: $g(x, y, z) = 2x + 2y + z - 108 = 0$.

$$\nabla V = yx\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}; \qquad \nabla g = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$$

$$\nabla V = \lambda \nabla g \implies yz = 2\lambda,\ xz = 2\lambda,\ xy = \lambda$$

Solve the system of equations:

$$yz = 2\lambda$$

$$xz = 2\lambda$$

$$xy = \lambda$$

$$2x + 2y + z = 108 \text{ (constraint equation)}$$

The solution is: $x = 18,\ y = 18,\ z = 36$. The dimensions that will maximize the volume of the box are: $18 \times 18 \times 36$; the maximum volume is: $V = 11,664$ cubic inches or 6.75 cubic feet.

(b) Let the dimensions of the box be: length – x, width – y, height – z.

The cost of construction is: $C(x, y, z) = 4(xy) + 3(2xz) + 3(2yz) = 4xy + 6xz + 6yz$

Minimize the cost: $C = 4xy + 6xz + 6yz$ subject to the constraint: $V(x, y, z) = xyz - 12 = 0$.

$$\nabla C = (4y + 6z)\,\mathbf{i} + (4x + 6z)\,\mathbf{j} + (6x + 6y)\,\mathbf{k}; \qquad \nabla V = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

$$\nabla C = \lambda \nabla V \implies 4y + 6z = \lambda yz,\ 4x + 6z = \lambda xz,\ 6x + 6y = \lambda xy$$

Solve the system of equations:

$$4y + 6z = \lambda yz$$

$$4x + 6z = \lambda xz$$

$$6x + 6y = \lambda xy$$

$$xyz = 12 \text{ (constraint equation)}$$

The solution is: $x = \sqrt[3]{18},\ y = \sqrt[3]{18},\ z = \sqrt[3]{16/3}$. The dimensions that will minimize the construction cost of the box are: $\sqrt[3]{18} \times \sqrt[3]{18} \times \sqrt[3]{16/3}$ feet.