

Is \mathbf{F} a gradient?

In each of the following, determine whether the given vector function \mathbf{F} is a gradient of a function f . If it is, find such an f .

1. $\mathbf{F}(x, y) = [ye^x(x+1) - 4e^{-2x}] \mathbf{i} + (xe^x - 2e^{2y} + 3) \mathbf{j}$.

Answer: Yes; $f(x, y) = xye^x - e^{2y} + 3y + 2e^{-2x}$.

2. $\mathbf{F}(x, y) = (\cos 2x + 3x^2y) \mathbf{i} + (x^3 - x + \ln y) \mathbf{j}$.

Answer: No.

3. $\mathbf{F}(x, y) = (\cos 2x - y + 3x^2y) \mathbf{i} + (x^3 - x + \ln y) \mathbf{j}$.

Answer: Yes; $f(x, y) = \frac{1}{2} \sin 2x - xy + x^3y + y \ln y - y$.

Multiple Integrals

1. Evaluate $\int \int_R 2y\sqrt{x} \, dx \, dy$ where R is the region in the first quadrant bounded by $y = x^2$, $y = 2 - x^2$, and the y -axis.

Answer: $\frac{32}{21}$

2. Evaluate $\int_0^9 \int_{\sqrt{y}}^3 \sin(\pi x^3) \, dx \, dy$

Answer: $\frac{2}{3\pi}$

3. Find the volume of the solid that is bounded above by the paraboloid $z = 8 - x^2 - y^2$, below by the x, y -plane, and on the sides by the cylinder $x^2 + y^2 = 4$, the plane $\sqrt{3}y = x$, and the y, z -plane.

Answer: 4π

4. Given $\int_{\Omega} \int (xy^2 - 2xy) \, dx \, dy$ where Ω is the region in the first quadrant bounded by $y = \sqrt{x}$, $x + y = 2$, and the x -axis.

(a) Set up a repeated integral (or integrals) integrating first with respect to x .

Answer: $\int_0^1 \int_{y^2}^{2-y} (xy^2 - 2xy) \, dx \, dy$.

(b) Set up a repeated integral (or integrals) integrating first with respect to y .

Answer: $\int_0^1 \int_0^{\sqrt{x}} (xy^2 - 2xy) dy dx + \int_1^2 \int_0^{2-x} (xy^2 - 2xy) dy dx.$

5. Evaluate $\int_{-1}^1 \int_0^x \int_0^{xy} (6z + 3y) dz dy dx.$

Answer: $\frac{2}{5}$

6. Let S be the solid in the first octant bounded by the cylinder $y^2 + z^2 = 1$ and the planes $y = x$, $x = 0$, and $z = 0$. Evaluate $\int \int \int_S 6z dx dy dz.$

Answer: $\frac{3}{4}$

7. Use spherical coordinates to find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = \sqrt{3x^2 + 3y^2}.$

Answer: $\frac{8\pi}{3} [2 - \sqrt{3}]$

8. Set up a repeated integral that gives the volume of the solid in the first octant whose top is the plane $x - y + 2z = 2$, and whose base in the x, y -plane is the region bounded by $y = \frac{1}{4}x^2$, $y = 4$, and the y -axis.

Answer:

$$\int_{\Omega} \int \frac{1}{2}[2 - x + y] dx dy = \int_0^4 \int_0^{2\sqrt{y}} \frac{1}{2}[2 - x + y] dx dy = \int_0^4 \int_{x^2/4}^4 \frac{1}{2}[2 - x + y] dy dx.$$

9. Set up a repeated integral in cylindrical coordinates that gives the volume of the solid in the first octant which is bounded above by the surface $z = e^{-x^2 - y^2}$, below by the x, y -plane, and on the sides by the cylinder $x^2 + y^2 = 1$.

Answer: $\int_0^{\pi/2} \int_0^1 \int_0^{e^{-r^2}} r dz dr d\theta$

10. A solid is bounded above by the surfaces $z = \sqrt{y}$ and $y + z = 2$, below by the x, y -plane, and on the sides by the planes $x = 0$ and $x = 2$. Find the volume of the solid.

Answer: $\frac{7}{3}$

11. The repeated integral $\int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-r^2}} r dz dr d\theta$ gives the volume of a solid T . Sketch T and evaluate the repeated integral.

Answer: $\frac{\pi}{12}$

12. The repeated integral $\int_{\pi/6}^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ gives the volume of a solid T . Sketch T and evaluate the repeated integral.

Answer: 3π

Line Integrals

1. Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ where \mathbf{h} is the vector field: $\mathbf{h}(x, y) = (x + 2y)\mathbf{i} + (x - 2y)\mathbf{j}$ and C is the line segment from $(1, 1)$ to $(3, -1)$

Answer: 0

2. Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ where \mathbf{h} is the vector field: $\mathbf{h}(x, y, z) = xz\mathbf{i} + (y + z)\mathbf{j} + x\mathbf{k}$, and C is the curve: $\mathbf{r}(u) = e^u\mathbf{i} + e^{-u}\mathbf{j} + e^{2u}\mathbf{k}$, $0 \leq u \leq \ln 2$.

Answer: $\frac{169}{24}$

3. Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r}$ where \mathbf{h} is the vector field: $\mathbf{h}(x, y) = (y^2 + 2xy)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$ and C is the line segment from $(-1, 2)$ to $(3, 1)$

Answer: 14

4. Let $f(x, y, z) = xy^3 + x^2z - x + z$. Calculate $\int_C \nabla f \cdot d\mathbf{r}$ where C is the curve: $\mathbf{r}(u) = 2 \cos u\mathbf{i} + 3 \sin u\mathbf{j} + 4\mathbf{k}$, $0 \leq u \leq \pi$.

Answer: 4

5. Find the work done by the force $\mathbf{F}(x, y) = x(y + 1)\mathbf{i} - 2xy\mathbf{j}$ applied to an object that moves along the line segments that connect $(0, 0)$, $(1, 0)$, $(2, 1)$ in that order.

Answer: $\frac{7}{6}$

6. Evaluate $\int_C -xy^2 dx + x^2y dy$ where C is the boundary of the triangular region with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$ traversed in the counterclockwise direction.

Answer: $\frac{2}{3}$

7. Evaluate $\int_C (2xe^y - x^2) dx + (x^2e^y + y) dy$ where C is the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ oriented counterclockwise.

Answer: 0

8. Evaluate the line integral $\int -x^2y dx + xy^2 dy$ where C is the boundary of the annular region in the first quadrant bounded by the quarter circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented counterclockwise.

Answer: 10π

9. Use Green's theorem to find the area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ Hint: parametrize the curve by $x = a \cos^3 u$, $y = a \sin^3 u$, $0 \leq u \leq 2\pi$.

Answer: $A = \frac{3}{8} \pi a^2$