Problem 1.

(a) The points $(3, -1, 2)$ and $(-1, 3, -4)$ are the endpoints of a diameter of a sphere.

(i) Determine the center and radius of the sphere.

(ii) Find an equation for the sphere.

Answer: (i) $C : (1, 1, -1);\ r = \sqrt{17}$ (ii) $(x-1)^2 + (y-1)^2 + (x+1)^2 = 17$

(b) Given the vectors $\mathbf{a} = 2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k},\ \mathbf{b} = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k},\ \mathbf{c} = \mathbf{i} + 2\,\mathbf{k}$.

(i) Calculate $2\mathbf{a} \cdot (\mathbf{b} - 3\mathbf{c})$.

(ii) Determine the vector projection of $\mathbf{c}$ onto $\mathbf{b}$.

(iii) Find the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

(iv) Find a unit vector that is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{c}$.

Answer:

(i) -32

(ii) $\frac{1}{14}(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$

(iii) $\dfrac{2}{3\sqrt{14}}$

(iv) $-\frac{2}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$

Problem 2. Given the planes $P_1 :\ 2(x-1) - (y+1) - 2(z-2) = 0,\ P_2 :\ 4x - 2y + 5z = 3,$ and the point $Q :\ (-2, 7, 4)$.

(a) Determine whether P_1 and P_2 are parallel, coincident, perpendicular, or none of the preceding.

Answer: $\mathbf{N}_1 \cdot \mathbf{N}_2 = 0$ implies $P_1 \perp P_2$

(b) Find an equation for the plane through Q which is parallel to P_1.

Answer: $2(x+2) - (y-7) - 2(z-4) = 0$

(c) Determine scalar parametric equations for the line through Q which is parallel to the line of intersection of P_1 and P_2.

Answer: $x = -2 - 9t, \quad y = 7 - 18t, \quad z = 4$

Problem 3. The position of an object at time t is given by: $\mathbf{r}(t) = e^{-t}\,\mathbf{i} + e^{t}\,\mathbf{j} - t\sqrt{2}\,\mathbf{k},\ 0 \le t < \infty$.

(a) Determine the velocity $\mathbf{v}$ and the speed of the object at time t.

Answer: velocity: $\mathbf{v} = -e^{-t}\,\mathbf{i} + e^{t}\,\mathbf{j} - \sqrt{2}\,\mathbf{k}$; speed: $\|\mathbf{v}\| = e^{t} + e^{-t}$

(b) Determine the acceleration of the object at time t.

Answer: acceleration: $\mathbf{a} = e^{-t}\,\mathbf{i} + e^{t}\,\mathbf{j}$

(c) Find the distance that the object travels during the time interval $0 \le t < \ln 3$.

Answer: $\dfrac{8}{3}$

Problem 4.

(a) A curve C in the plane is defined by the parametric equations: $x = t^2 + 1,\ y = \frac{4}{3}t^3 - 1$.

(i) Find the length of C from $t = 0$ to $t = 2$.

(ii) Find the curvature of C at $t = 1$.

Answer: (i) $\frac{1}{6}\left([17]^{3/2} - 1\right)$ units (ii) $\kappa = \dfrac{\sqrt{5}}{25}$

(b) The vector function $\mathbf{r}(t) = \sin 2t\,\mathbf{i} - \cos 2t\,\mathbf{j} + t\sqrt{5}\,\mathbf{k}$ determines a curve C in space.

(i) Find the unit tangent vector $\mathbf{T}$ and the principal unit normal $\mathbf{N}$.

(ii) Determine the curvature of C at time t.

(iii) Determine the tangential and normal component of the acceleration vector.

Answer: (i) $\mathbf{T} = \frac{2}{3}\cos 2t\,\mathbf{i} + \frac{2}{3}\sin 2t\,\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$; $\mathbf{N} = -\sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j}$

(ii) $\dfrac{4}{9}$ (iii) $a_{\mathbf{T}} = 0, \quad a_{\mathbf{N}} = 4$

Problem 5. Let $f(x, y) = x\ln(x/y) + xy^2$.

(a) Calculate f_{xx} and f_{yz}.

Answer: $f_{xx} = \dfrac{1}{x}, \qquad f_{yx} = 2y - \dfrac{1}{y}$

(b) Determine the directional derivative of f at the point $(2, 2)$ in the direction of the vector $\mathbf{a} = \mathbf{i} - 2\,\mathbf{j}$.

Answer: $-\dfrac{9}{\sqrt{5}}$

(c) Suppose that $x = st\,e^t$ and $y = 2s\,e^t$. Calculate $\dfrac{\partial f}{\partial t}$.

Answer: $\dfrac{\partial f}{\partial t} = \left[1 + \ln(x/y) + y^2\right]\left[se^t + ste^t\right] + \left[-\dfrac{x}{y} + 2xy\right]2se^t$

(d) Determine an equation for the tangent plane to the surface $z = f(x, y)$ at the point $(2, 2, 8)$ on the surface.

Answer: $5(x - 2) + 7(y - 2) - (z - 8) = 0$

Problem 6. Let $F(x, y, z) = 2xy^2 + 2yz^2 + 2x^2z$.

(a) Determine the maximum directional derivative of F at the point $(1, -1, 1)$.

Answer: maximum directional derivative: $\|\nabla F\| = \sqrt{44}$

(b) Find the directional derivative of F at the point $(-2,1,-1)$ in the direction parallel to the line $x = 3+4t,\ y = 2-t,\ z = 3t$.

Answer: $\dfrac{58}{\sqrt{26}}$

(c) Determine symmetric equations for the normal line to the level surface $F(x,y,z) = -2$ at the point $(-1,2,1)$.

Answer: $x = -1+4t, \quad y = 2-6t, \quad z = 1+10t$

(d) Suppose the $x = t^2+1,\ y = 2t,\ z = t^3$. Calculate $\dfrac{dF}{dt}$.

Answer: $\dfrac{dF}{dt} = \left(2y^2+4xz\right)2t + \left(4xy+2z^2\right)2 + \left(4yz+2x^2\right)3t^2$

Problem 7.

(a) Let $f(x,y) = 3x^2y - 2y^2 - 3x^2 - 8y + 2$.

(i) Find the stationary points of f.

(ii) For each stationary point P found in (i), determine whether f has a local maximum, a local minimum, or a saddle point at P.

Answer: (i) $(0,-2),\ (2,1),\ (-2,1)$ (ii) $(0,-2)$ loc. max, $(\pm 2, 1)$ saddle points

(b) Determine the minimum value of $f(x,y) = 2x^2+xy-y^2+1$ subject to the constraint $2x+3y = 16$.

Answer: $f(-7,10) = -71$ minimum value

Problem 8.

(a) Find the absolute maximum and absolute minimum values of $f(x,y) = x^2+2y^2-2x+2$ on the closed disk $D:\ x^2+y^2 \le 4$.

Answer: absolute min: $f(1,0) = 1$; absolute max: $f(-1,\pm\sqrt{3}) = 11$

(b) Find the absolute maximum and absolute minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ on the closed triangular region bounded by the lines $x = 0,\ y = 0,\ x+y = 9$.

Answer: absolute min: $f(0,9) = f(9,0) = -61$; absolute max: $f(1,1) = 4$

Problem 9.

(a) An open-topped rectangular container is to have a volume of 32 cubic meters. Find the dimensions of the container having the smalllest surface area.

Answer: length – 4 meters, width – 4 meters, height – 2 meters

(b) The temperature T at a point (x, y, z) on the sphere $x^2+y^2+z^2 = 1$ is given by $T(x, y, z) = 400xyz^2$. What are the maximum and minimum temperatures?

Answer: max temp: $T\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = 50°$

min temp: $T\left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = T\left(\frac{1}{2}, -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = -50°$

Problem 10.

(a) Given the repeated integral $\int_0^2 \int_{x^2}^4 2x\,\cos(y^2)\,dy\,dx$. Determine an equivalent repeated integral with the order of integration reversed. Evaluate one of the two integrals.

Answer: $\int_0^2 \int_{x^2}^4 2x\cos(y^2)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} 2x\cos(y^2)\,dx\,dy = \frac{1}{2}\sin 16$

(b) Express the area of the region bounded by the curves $y = 2x$ and $y = x^2$ by a repeated integral integrating: (i) first with respect to y, then with respect to x; (ii) first with respect to x, then with respect to y.

Answer: $A = \int_0^2 \int_{x^2}^{2x} 1\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} 1\,dx\,dy = \dfrac{4}{3}$

(c) Use a double integral to find the volume of the solid S in the first octant that is bounded above by the surface $z = 4 - x^2 - y^2$, below by the x, y-plane, and on the sides by the planes $y = 0$ and $y = x$.

Answer: $V = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} (4 - x^2 - y^2)\,dy\,dx = \int_0^{\pi/4} \int_0^2 (4 - r^2)r\,dr\,d\theta = \pi$

Problem 11.

(a) Evaluate $\iiint_T 2xy\,dx\,dy\,dz$ where T is the solid in the first octant bounded above by the cylinder $z = 4 - x^2$ below by the x, y-plane, and on the sides by the planes $x = 0$, $y = 2x$ and $y = 4$.

Answer: $\int_0^4 \int_0^{y/2} \int_0^{4-x^2} 2xy\,dz\,dx\,dy = \int_0^2 \int_{2x}^4 \int_0^{4-x^2} 2xy\,dz\,dy\,dx = \dfrac{128}{3}$

(b) Set up a triple integral in cylindrical coordinates that gives the volume of the solid in the first octant that is bounded above by the hemisphere $z = \sqrt{2 - x^2 - y^2}$, below by the paraboloid $z = x^2 + y^2$ and on the sides by the xz- and yz-planes.

Answer: $V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} 1\,dz\,dy\,dx = \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$

(c) Set up a triple integral in spherical coordinates that gives the volume of the solid that lies outside the cone $z = \sqrt{x^2 + y^2}$ and inside the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Answer: $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Problem 12.

(a) Let $\mathbf{h}(x, y, z) = xy\,\mathbf{i} + y\,\mathbf{j} - yz\,\mathbf{k}$, and let C be the curve given by $\mathbf{r}(u) = u\,\mathbf{i} + u^2\,\mathbf{j} + 2u\,\mathbf{k}$, $0 \leq u \leq 1$. Calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr}$.

Answer: $-\dfrac{1}{4}$

(b) Show that $\mathbf{h}(x, y) = (6xy - y^2)\,\mathbf{i} + (4y + 3x^2 - 2xy)\,\mathbf{j}$ is the gradient of a function f. Use this information to calculate $\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr}$ where C is the curve consisting of the line segment from $(0, 0)$ to $(2, 4)$ and the parabola $y = x^2$ from $(2, 4)$ to $(3, 9)$.

Answer: 162

Problem 13.

(a) Let $\mathbf{h}(x, y) = 2xy^3\,\mathbf{i} + 4x^2y^2\,\mathbf{j}$. Calculate $\oint_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr}$ where C is the boundary of the triangular region in the first quadrant bounded by the x-axis, the line $x = 1$ and the curve $y = x^3$.

Answer: $\dfrac{2}{33}$

(b) Let $\mathbf{g}(x, y) = (2xy + e^x - 3)\,\mathbf{i} + (x^2 - y^2 + \sin y)\,\mathbf{j}$. Calculate $\oint_C \mathbf{g}(\mathbf{r}) \cdot \mathbf{dr}$ where C is the ellipse $4x^2 + 9y^2 = 36$. Hint: check to see if $\mathbf{g}$ is a gradient.

Answer: 0

(c) Use Green's Theorem to find the area enclosed by the astroid $\mathbf{r}(u) = \cos^3 u\,\mathbf{i} + \sin^3 u\,\mathbf{j}$, $0 \leq u \leq 2\pi$.

Answer: $\dfrac{3\pi}{8}$