DIFFERENTIAL EQUATIONS

Background material: Text, Section 1.1

1. BASIC TERMINOLOGY (Text, Section 1.2)

A **differential equation** is an equation that contains an unknown function together with one or more of its derivatives.

Examples: In 1 − 4 , find $y(x)$ such that:

1. $y' = 2x + \cos x$

2. $\dfrac{dy}{dx} = ky$ (exponential growth/decay)

3. $x^2 y'' - 2xy' + 2y = 4x^3$

4. $$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$$

Find $u(x, y)$ such that

5. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (Laplace's eqn.)}$$

TYPE:

If the unknown function depends on a single independent variable, then the equation is an

ordinary differential equation (ODE).

If the unknown function depends on more than one independent variable, then the equation is a

partial differential equation (PDE).

ORDER:

The **order** of a differential equation is the order of the **highest derivative of the unknown function** appearing in the equation.

Examples:

1. $y' = 2x + \cos x$

Type: Order:

2. $\dfrac{dy}{dx} = ky$ (exponential growth/decay)

Type: Order:

3. $x^2y'' - 2xy' + 2y = 4x^3$

Type: Order:

4. $$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$$

Type: Order:

5. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (Laplace's eqn.)}$$

Type: Order:

6. $$\frac{d^2y}{dx^2} + 2x \sin\left(\frac{dy}{dx}\right) + 3e^{xy} = \frac{d^3}{dx^3}(e^{2x})$$

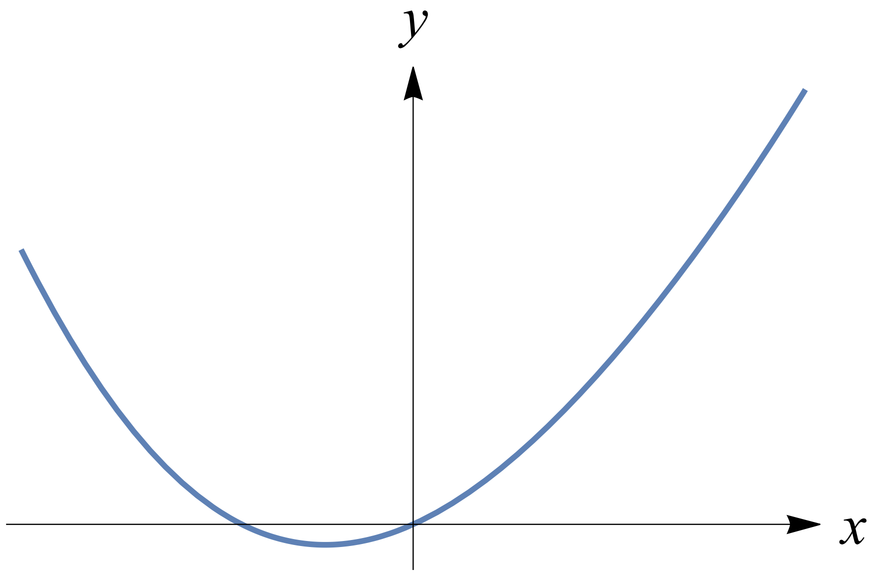
Type: Order:

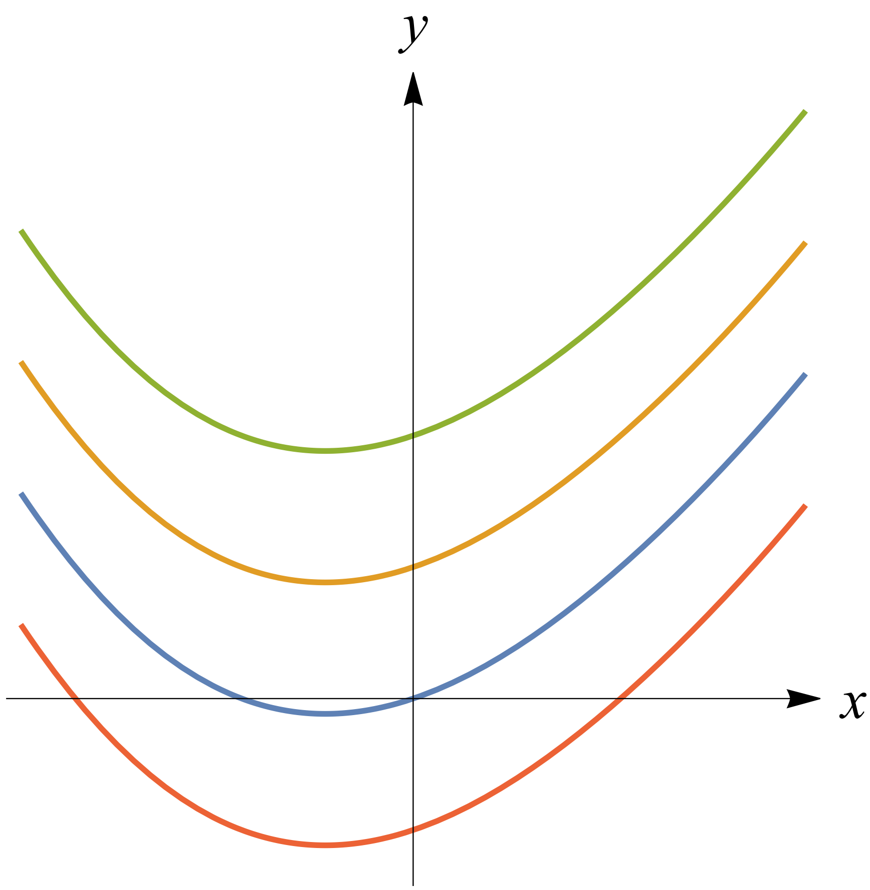
2. SOLUTIONS OF DIFFERENTIAL EQUATIONS

A **solution of a differential equation** is a function defined on some domain D such that the equation reduces to an identity when the function is substituted into the equation.

Examples:

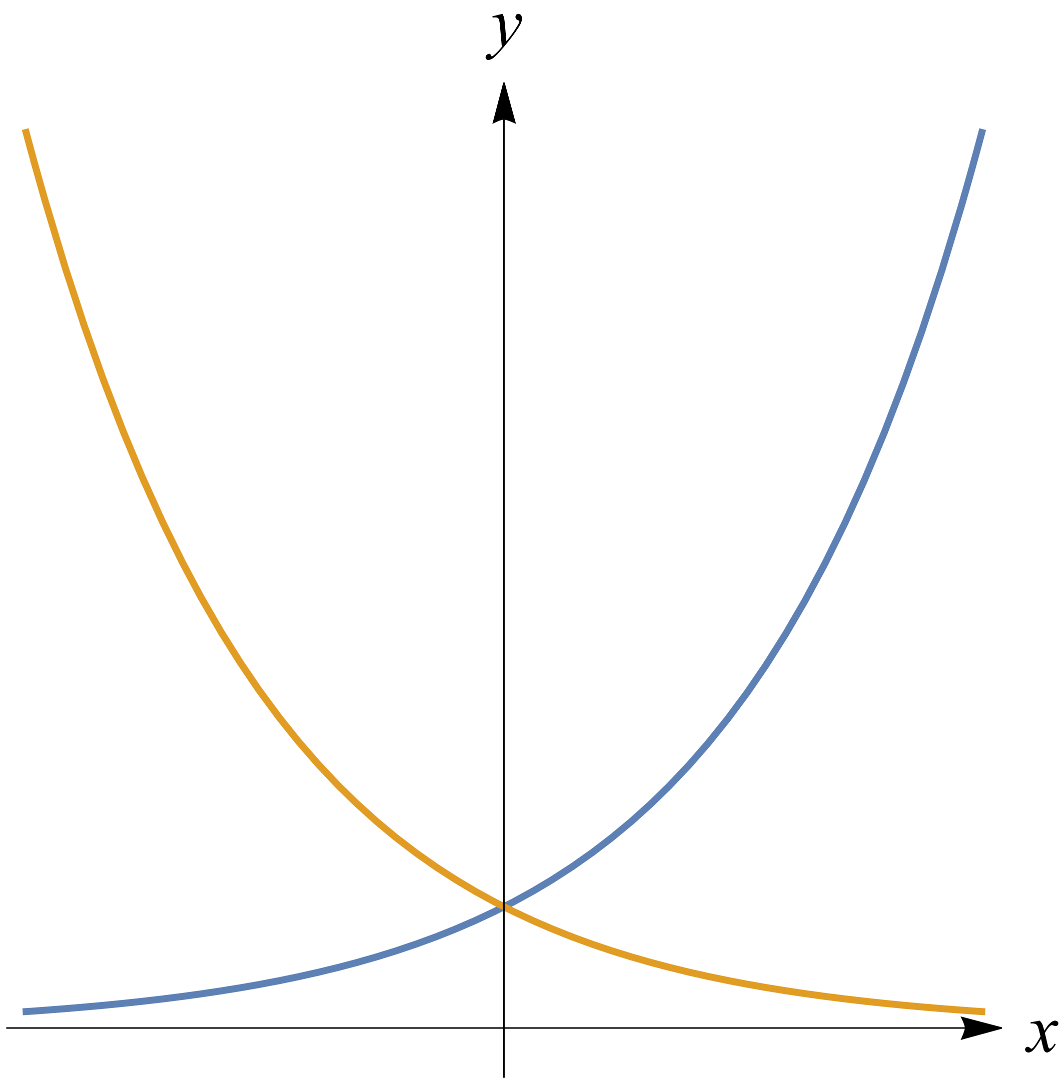
1. $y' = 2x + \cos\, x$


y
x


y
x

2. $y' = ky$

$$y = e^{kx}$$



3. $y'' - 2y' - 8y = 4e^{2x}$

Is $y = 2e^{4x} - \frac{1}{2}e^{2x}$ a solution?

$$y'' - 2y' - 8y = 4e^{2x}$$

Is $y = e^{-2x} + 2e^{3x}$ a solution?

4. $x^2y'' - 4xy' + 6y = 3x^4$

Is $y = \frac{3}{2}x^4 + 2x^3$ a solution?

$$x^2y'' - 4xy' + 6y = 3x^4$$

Is $y = 2x^2 + x^3$ a solution?

5. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u = \ln\sqrt{x^2 + y^2}$ Solution?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = \cos x \sinh y, \qquad u = 3x - 4y$$

Solutions??

Finding solutions - simple equations, from calculus

1. Find the solutions of:

$y' = 6x^2 + 4\cos 2x$

2. Find the solutions of:

$y'' = 6e^{3x} + 12x$

Finding solutions - with hints

1. Find a value of r, if possible, such that $y = e^{rx}$ is a solution of:

(a) $y'' - 3y' - 10y = 0$

(b) $y'' - 6y' + 13y = 0$

2. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$x^2 y'' + 2x\, y' - 6y = 0.$$

3. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$y'' - \frac{1}{x}\,y' - \frac{3}{x^2}\,y = 0.$$

3. n-PARAMETER FAMILY OF SOLUTIONS / GENERAL SOLUTION (Text, Section 1.3)

Example: Find solutions of the differential equation:

$$y''' - 12x + 6e^{2x} = 0$$

NOTE: To solve a differential equation having the special form

$$y^{(n)}(x) = f(x),$$

simply integrate f n times,

and EACH integration step produces an arbitrary constant;

there will be n independent arbitrary constants.

Intuitively, to find a set of solutions of an n-th order differential equation

$$F\left[x, y, y', y'', \ldots, y^{(n)}\right] = 0$$

we "integrate" n times, with each integration step producing an arbitrary constant of integration (i.e., a **parameter**). Thus, "in theory," an n-th order differential equation has an **n-parameter family of solutions**.

SOLVING A DIFFERENTIAL EQUATION:

To **solve** an n-th order differential equation

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

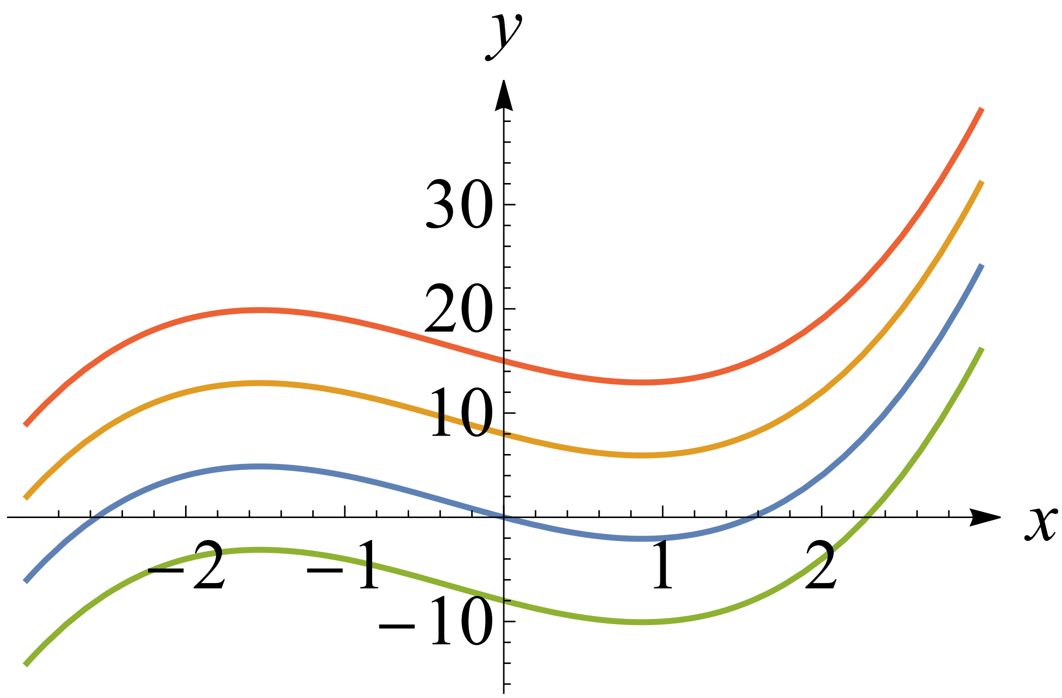
means to find an n-parameter family of solutions. (**Note:** Same n.)

NOTE: An "n-parameter family of solutions" is more commonly called **THE GENERAL SOLUTION**.

Examples: Find the general solution:

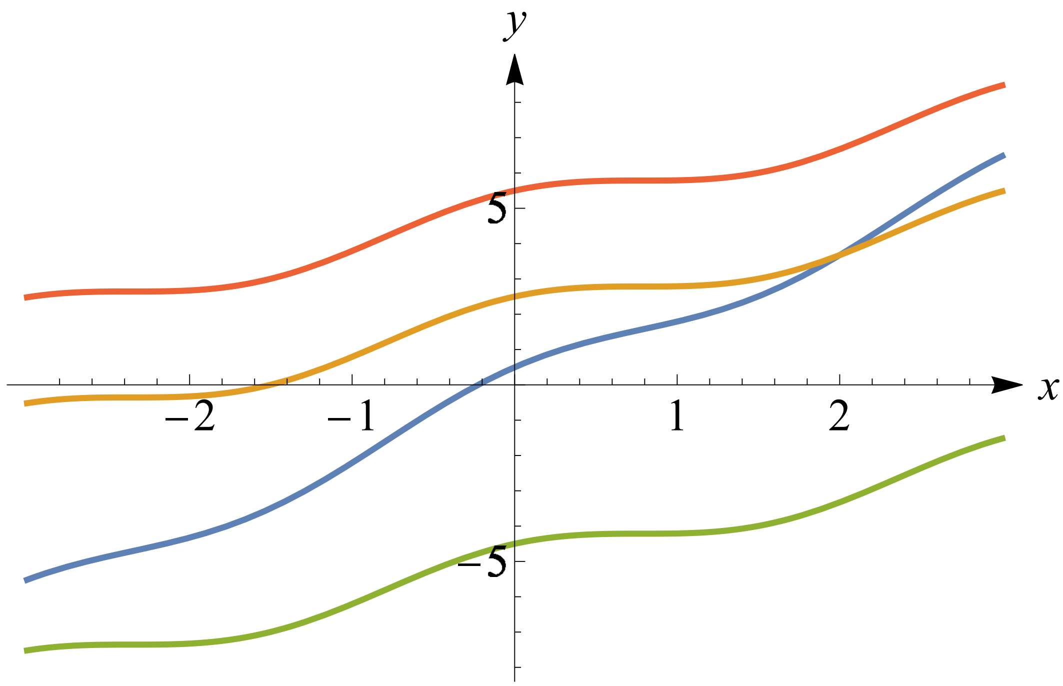
1. $y' = 3x^2 + 2x - 4$

$$y = x^3 + x^2 - 4x + C$$



2. $y'' + 2\sin 2x = 0$

$$y = \frac{1}{2}\sin 2x + C_1 x + C_2$$



3. $y''' - 3y'' + 3y' - y = 0$

Answer: $y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$

4. $x^2 y'' - 4xy' + 6y = 3x^4$

Answer: $y = C_1 x^2 + C_2 x^3 + \frac{3}{2} x^4$

PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a **particular solution**.

Examples:

1. $y'' = 6x + 8e^{2x}$

General solution:

$$y = x^3 + 2e^{2x} + C_1x + C_2$$

Particular solutions:

2. $x^2y'' - 2xy' + 2y = 4x^3$

General solution:

$$y = C_1x + C_2x^2 + 2x^3$$

Particular solutions:

4. THE DIFFERENTIAL EQUATION OF AN n-PARAMETER FAMILY:

Given an n-parameter family of curves.

The **differential equation of the family** is an n-th order differential equation that has the given family as its general solution.

Examples:

1. $y^2 = Cx^3 + 4$ is the general solution of a DE.

a. What is the order of the DE?

b. Find the DE.

2. $y = C_1 x + C_2 x^3$ is the general solution of a DE.

a. What is the order of the DE?

b. Find the DE?

General strategy for finding the differential equation of an n-parameter family

Step 1. Differentiate the family n times. This produces a system of $n+1$ equations.

Step 2. Choose any n of the equations and solve for the parameters.

Step 3. Substitute the "values" for the parameters in the remaining equation.

Examples:

The given family of functions is the general solution of a differential equation.

(a) What is the order of the equation?

(b) Find the equation.

1. $y = Cx^3 - 2x$

(a)

(b)

2. $y = C_1 e^{2x} + C_2 e^{3x}$

(a)

(b)

3. $y = C_1 \cos 3x + C_2 \sin 3x$

(a)

(b)

4. $y = C_1x^4 + C_2x + C_3$

(a)

(b)

5. $y = C_1 + C_2x + C_3x^2$

(a)

(b)

5. INITIAL-VALUE PROBLEMS:

(Text, Section 1.4)

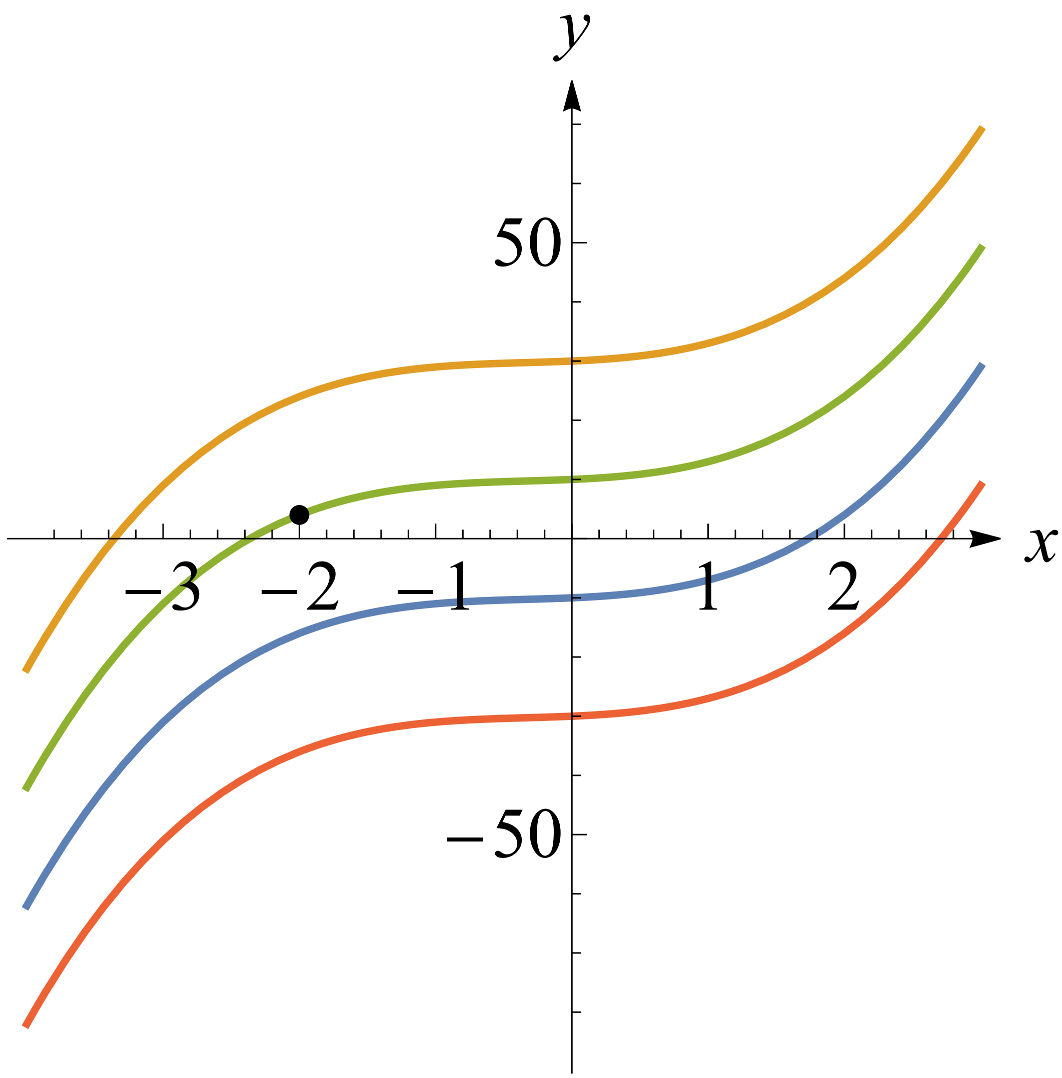
1. Find a solution of

$y' = 3x^2 + 2x + 1$

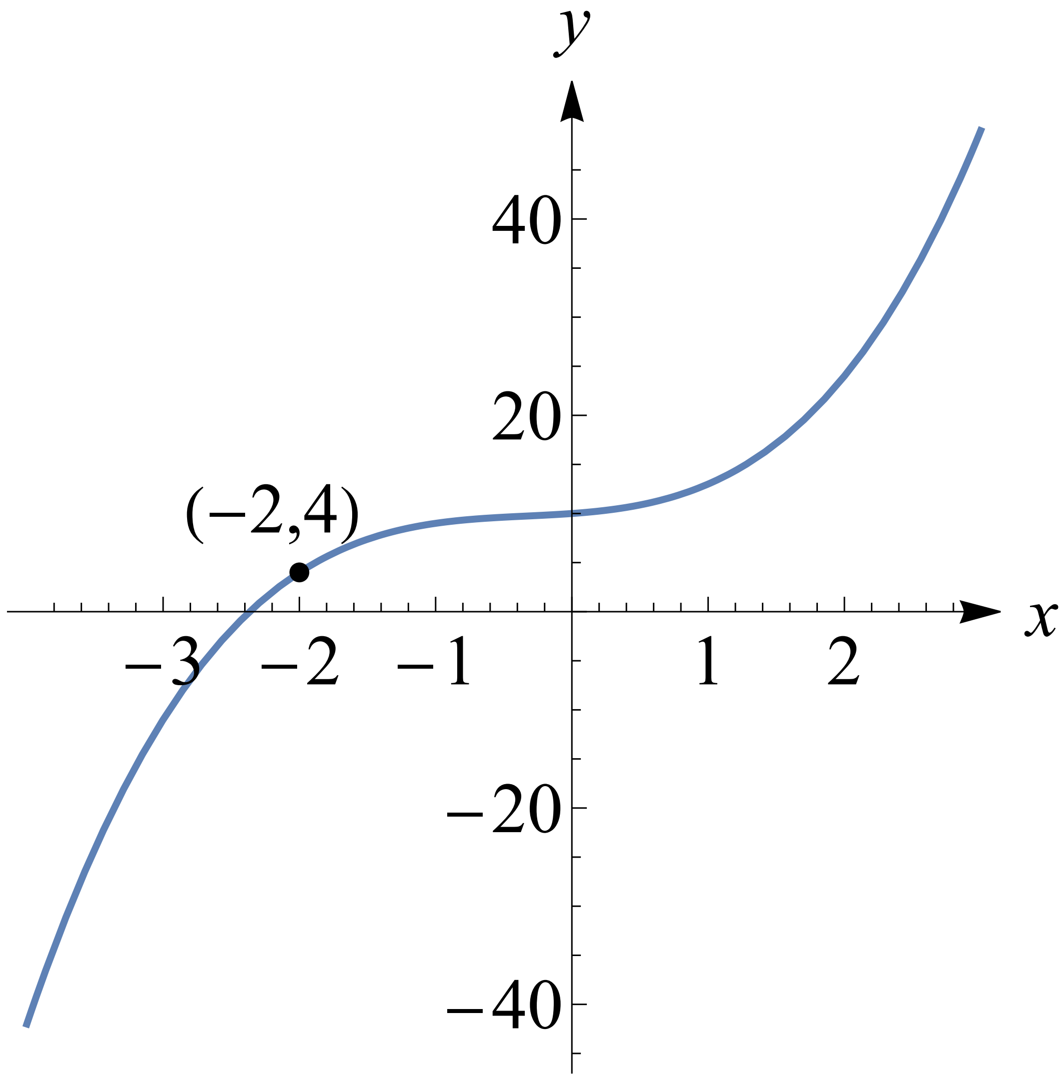
which passes through the point $(-2, 4)$;

that is, satisfies $y(-2) = 4$.

$y = x^3 + x^2 + x + C$ (the general solution)



$y = x^3 + x^2 + x + 10$ (the particular solution that satisfies the equation)



2. $y = C_1 \cos 3x + C_2 \sin 3x$ is the general solution of:

$$y'' + 9y = 0.$$

a. Find a solution that satisfies

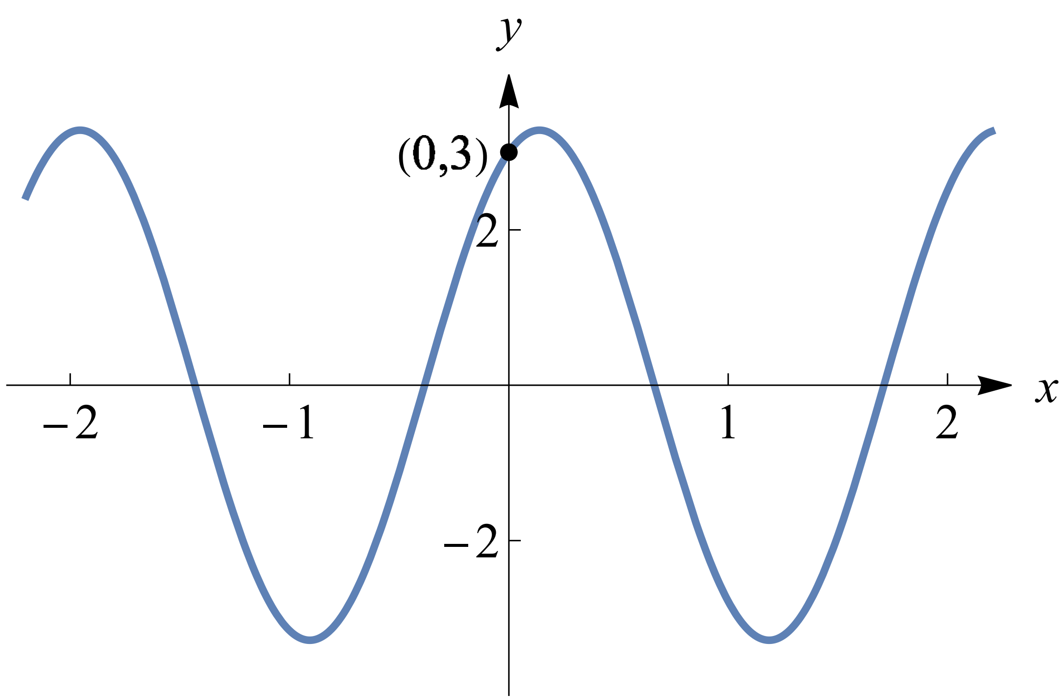
$y(0) = 3$

b. Find a solution which satisfies

$y(0) = 3,\ y'(0) = 4$

$y = 3\cos 3x + \frac{4}{3}\sin 3x$ is the solution of

$$y'' + 9y = 0, \quad y(0) = 3,\ y'(0) = 4.$$



An n**-th order initial-value problem** consists of an n-th order differential equation

$$F\left[x, y, y', y'', \ldots, y^{(n)}\right] = 0$$

together with n (initial) conditions of the form

$$y(c) = k_0,\ \ y'(c) = k_1,\ \ y''(c) = k_2,\ \ \ldots,$$

$$y^{(n-1)}(c) = k_{n-1}$$

where c and $k_0,\ k_1,\ \ldots,\ k_{n-1}$ are given numbers.

NOTES:

1. An n-th order differential equation can always be written in the form

$$F\left[x, y, y', y'', \cdots, y^{(n)}\right] = 0$$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.

Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

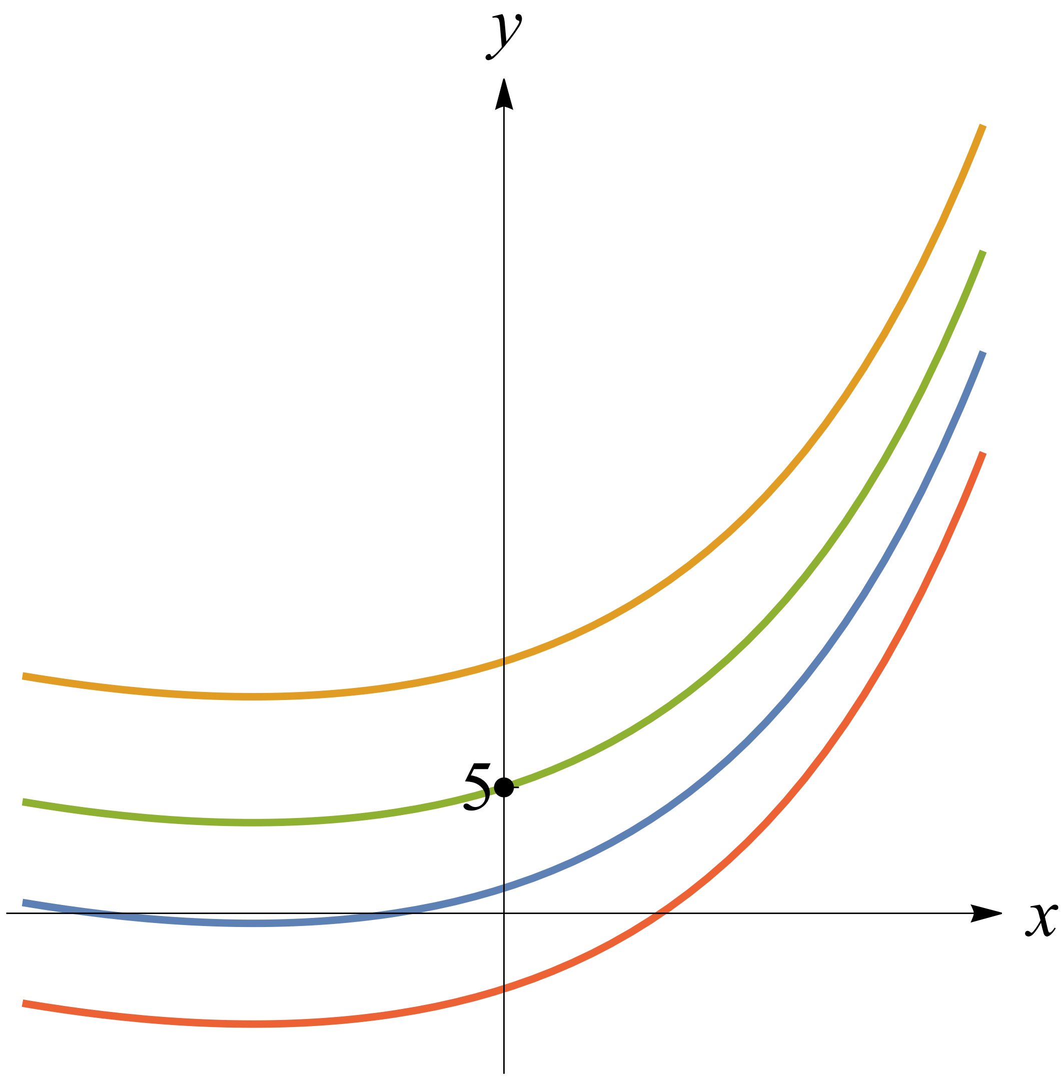
Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.

Examples:

1. Find a solution of the initial-value problem

$$y' = 4x + 6e^{2x},\ \ y(0) = 5$$

General solution: $y = 2x^2 + 3e^{2x} + C$



2. $y = C_1 e^{-2x} + C_2 e^{4x}$ is the general solution of

$$y'' - 2y' - 8y = 0$$

Find a solution that satisfies the initial conditions

$$y(0) = 3, \; y'(0) = 2$$

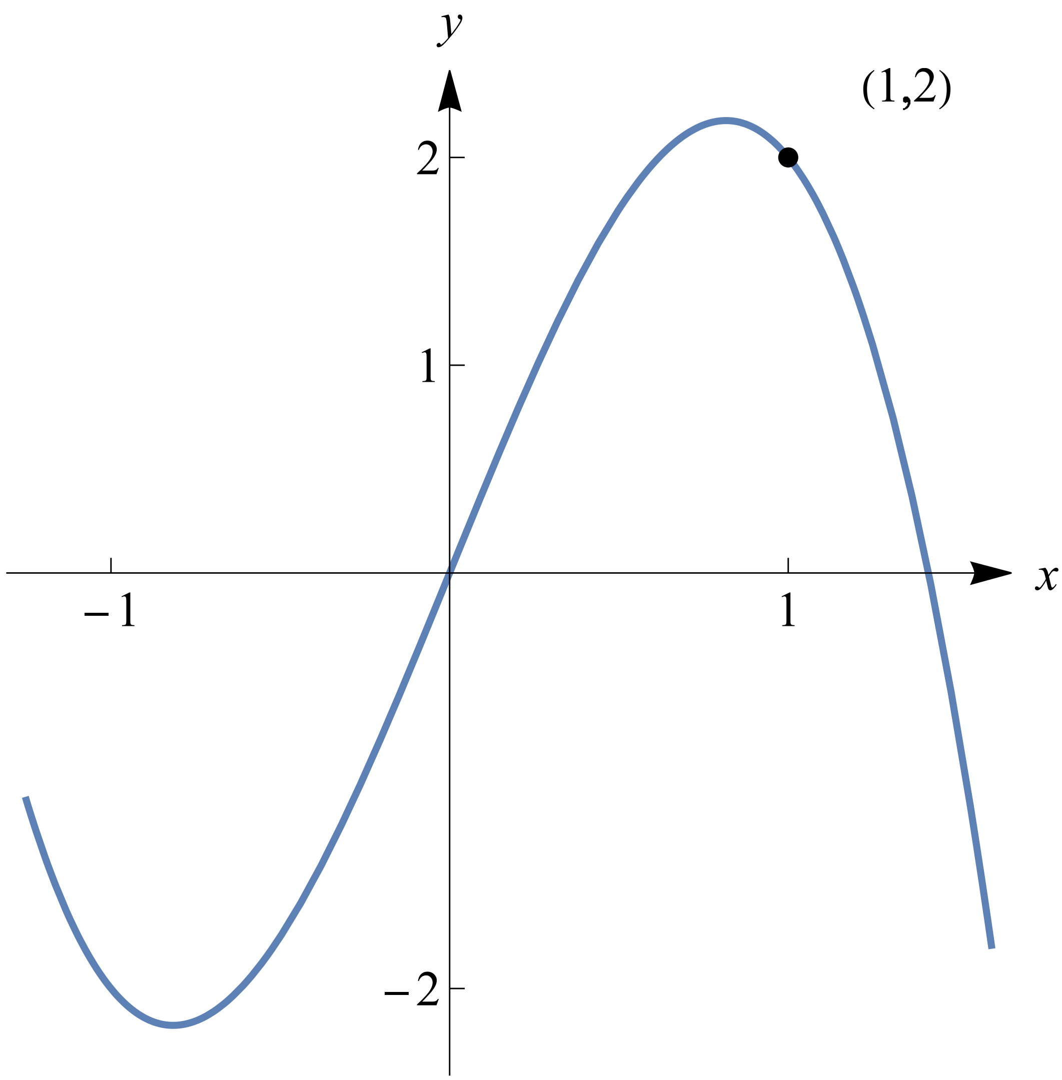
3. $y = C_1 x + C_2 x^3$ is the general solution of

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0$$

a. Find a solution which satisfies

$$y(1) = 2, \quad y'(1) = -2.$$

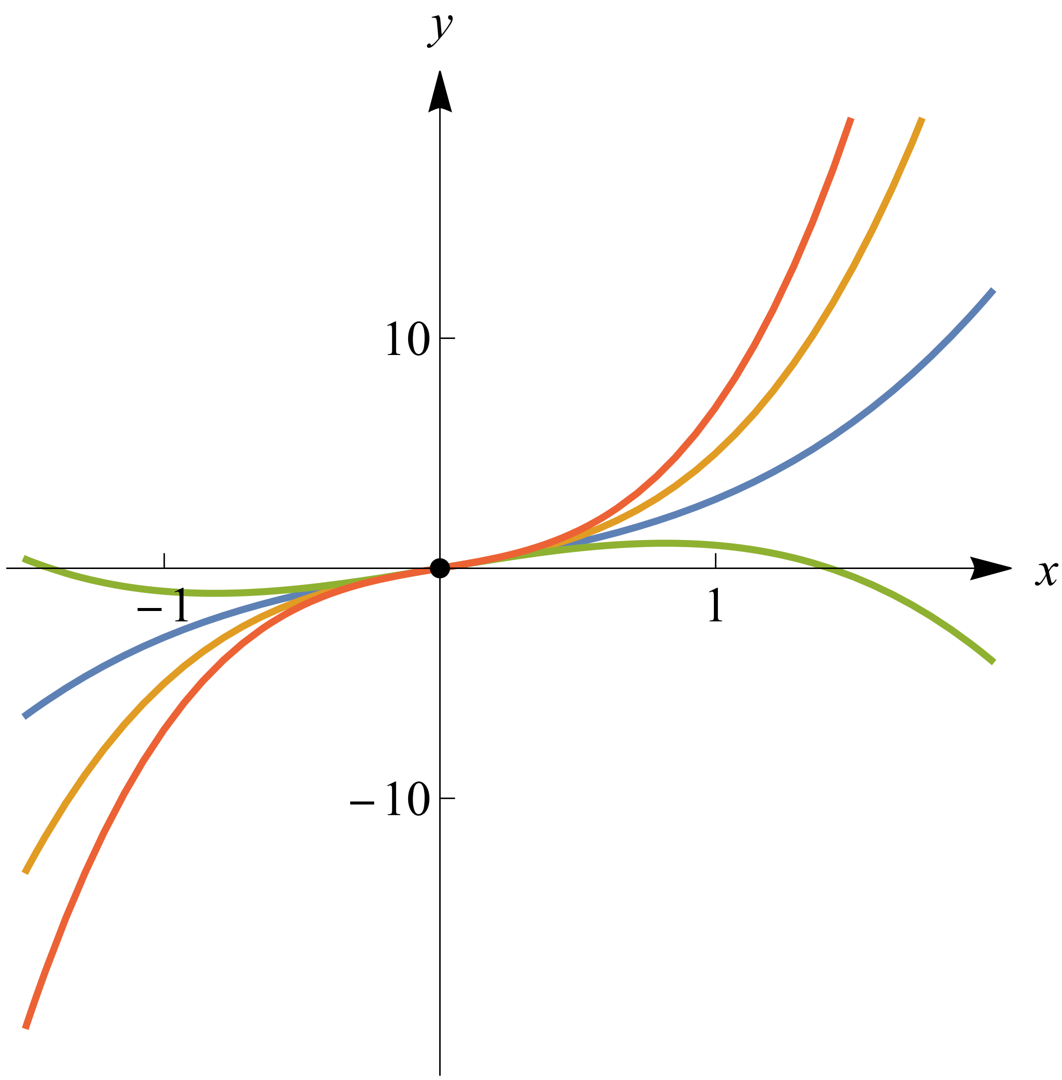
Graph: $y = 4x - 2x^3$



b. Find a solution which satisfies

$$y(0) = 0, \quad y'(0) = 2.$$

Graphs: $y = 2x + C_2x^3$



c. Find a solution which satisfies

$$y(0) = 2, \quad y'(0) = -2.$$

EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem **have** a solution? That is, do solutions to the problem **exist**?

2. If a solution does exist, is it **unique**? That is, is there exactly one solution to the problem or is there more than one solution?

Chapter 1. Terms

Differential Equation, pg. 6

Type, pg. 6

Order, pg. 7

Solution, pg. 7

n-Parameter Family of Solns, pg. 14

General Solution, pg. 16

Singular Solution, pg. 16

Particular Solution, pg. 17

Differential Equation of an n-Parameter Family, pg. 17

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