Chapter 2, Part 1

FIRST ORDER EQUATIONS

$$F(x, y, y') = 0$$

Background Material: (Text, Section 2.1)

Techniques of integration:

- Substitution (the most common technique)

- Integration-by-parts

- Integrals involving trig functions

- Partial fraction decomposition

Given a first order DE

$$F(x, y, y') = 0$$

Basic assumption: The equation can be solved for y'; that is, the equation can be written in the form

$$y' = f(x, y) \qquad (1)$$

2.2. FIRST ORDER LINEAR EQUATIONS (Text: Section 2.2)

$$y' = f(x, y)$$

is a **linear equation** if f has the form

$$f(x, y) = P(x)y + q(x)$$

where P and q are continuous functions on some interval I. Thus

$$y' = P(x)y + q(x)$$

Standard form:

The **standard form** for a first order linear equation is:

$$\boxed{y' + p(x)y = q(x)}$$

where p and q are continuous functions on the interval I

(**Note:** A differential equation which is not linear is called **nonlinear**.)

Examples:

1. Find the general solution:

$y' = ky,\ k$ constant (See Examples in Chapter 1)

2. Find the general solution:

$$y' + 2xy = 4x$$

Solution Method:

Step 1. **Identify**: Determine that the equation IS linear and write it in standard form

$$y' + p(x)y = q(x).$$

$$y' + p(x)y = q(x).$$

Step 2. Multiply by $e^{\int p(x)\,dx}$:

$$\left[e^{\int p(x)\,dx}\,y\right]' = q(x)e^{\int p(x)\,dx}$$

Step 3. Integrate:

$$e^{\int p(x)\,dx}\,y = \int q(x)e^{\int p(x)\,dx}\,dx + C.$$

Step 4. Solve for y:

$$y = e^{-\int p(x)\,dx}\int q(t)e^{\int p(t)\,dt}\,dx + Ce^{-\int p(x)\,dx}.$$

$$y = e^{-\int p(x)\,dx} \int q(x) e^{\int p(x)\,dx}\,dx + Ce^{-\int p(x)\,dx}$$

is the general solution of the equation.

Note: $e^{\int p(x)\,dx}$ is called an **integrating factor**

3. Find the general solution:

$$xy' = \frac{\cos 2x}{x^2} - 3y$$

4. Find the general solution:

$$xy' = \frac{2}{\sqrt{x^2 - 1}} - 2y + 2$$

5. Solve the initial-value problem:

$$y' + (\cot x)y = 2\cos x, \quad y(\pi/2) = 3$$

6. Find the general solution:

$$y' + 2xy = 2\tan x$$

Answers:

1. $y = Ce^{kx}$

2. $y = 2 + Ce^{-x^2}$

3. $y = \dfrac{\sin 2x}{2x^3} + \dfrac{C}{x^3}$

4. $y = \dfrac{2\sqrt{x^2 - 1}}{x^2} + 1 + \dfrac{C}{x^2}$

5. $y = \dfrac{5 - \cos 2x}{2 \sin x}$

6. $y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2}$

Linear Operations and the term "linear"

Differentiation:

As you know: For differentiable functions f and g

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{df}{dx}+\frac{dg}{dx}$$

and for any constant c

$$\frac{d}{dx}\left[c\,f(x)\right]=c\,\frac{df}{dx}$$

Integration:

For integrable functions f and g:

$$\int [f(x) + g(x)]\ dx = \int f(x)\, dx + \int g(x)\, dx$$

and, for any constant c

$$\int c\, f(x)\, dx = c \int f(x)\, dx$$

Any “operation” L which satisfies

$$L\,[f(x)+g(x)] = L[f(x)] + L[g(x)]$$

and $\quad L\,[c\,f(x)] = c\,L\,[f(x)]$

is a “linear” operation.

1. **Differentiation** is a linear operation.

2. **Integration** is a linear operation.

Set $L[y] = y' + p(x)y$

$$L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)$$

$$= y_1' + y_2' + py_1 + py_2$$

$$= y_1' + py_1 + y_2' + py_2 = L[y_1] + L[y_2]$$

$$L[cy] = (cy)' + p(cy) = cy' + cpy$$

$$= c(y' + py) = cL[y]$$

Thus, if $L[y] = y' + p(x)y$, then

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$

$$L[c\,y] = c\,L[y]$$

$L[y] = y' + p(x)y$: the left-hand side of a linear differential equation in standard form; L is a **linear operator.** Hence the term linear differential equation.

Example. Let $L[y] = y' + \dfrac{2}{x}y$.

(a) Find $L[2x^2 - 3x]$

(b) Find y such that $L[y] = \dfrac{e^{4x}}{x^2}$

2.3. SEPARABLE EQUATIONS

(Text, Section 2.3)

$$y' = f(x, y)$$

is a **separable equation** if f can be **factored** into

$$f(x, y) = p(x)h(y)$$

where p and h are continuous functions.

$$\boxed{y' = p(x)h(y)}$$

is called the "standard form."

Example 1: Show that

$$y' = xy^2 - x - y^2 + 1$$

is separable

Solution Method:

Step 1. **Identify**: Establish that the equation IS separable.

Step 2. Divide both sides by $h(y)$ to "separate" the variables.

$$\frac{1}{h(y)}y' = p(x) \quad \text{or} \quad q(y)y' = p(x)$$

which, can be written as

$$q(y)\,\frac{dy}{dx} = p(x) \quad \text{and} \quad q(y)dy = p(x)dx$$

the variables are "separated."

Step 3. Integrate

$$q(y)dy = p(x)dx$$

$$\int q(y)\, dy = \int p(x)\, dx + C$$

$$Q(y) = P(x) + C$$

where $Q'(y) = q(y),\ P'(x) = p(x)$

Note:

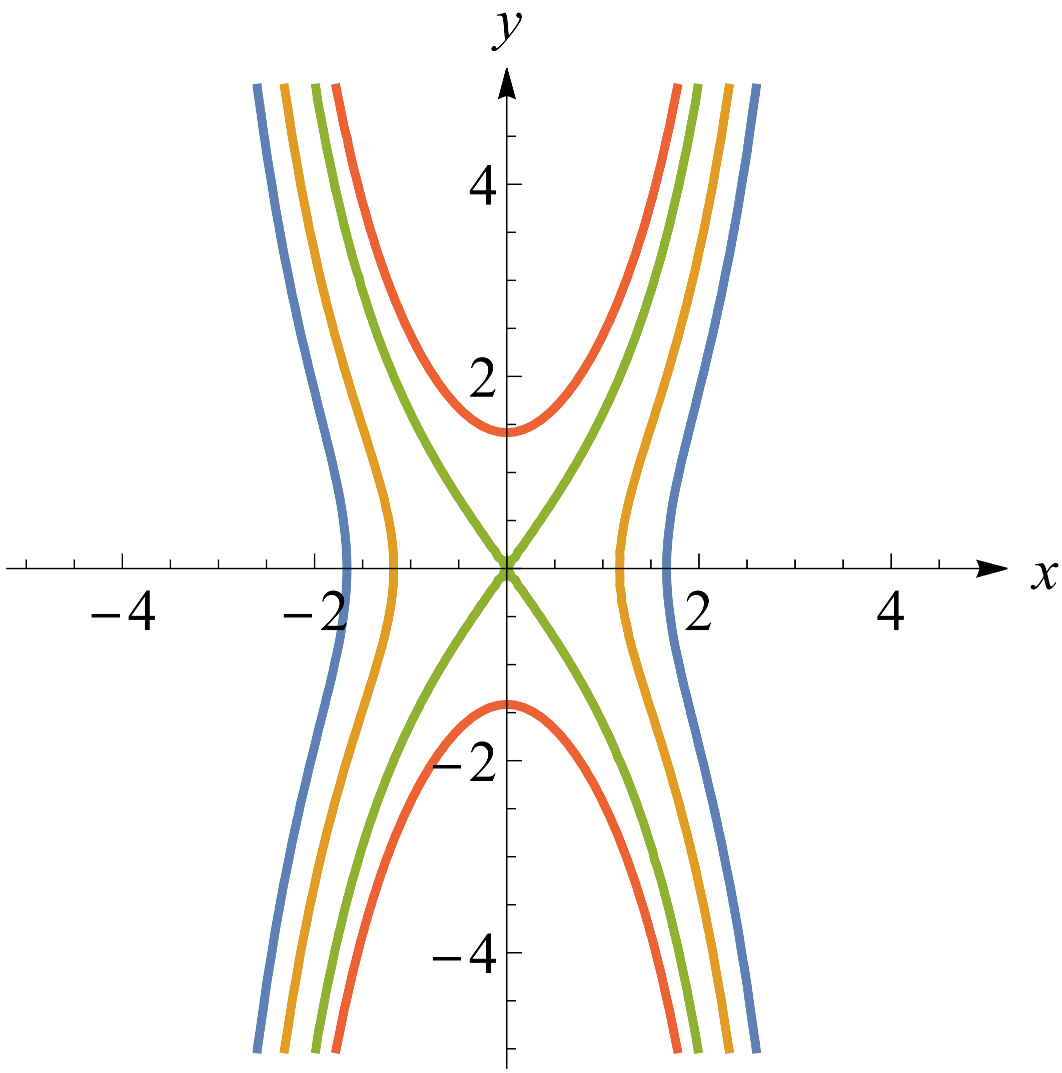
$Q(y) = P(x) + C$ is the general solution. Typically, this is an implicit relation between x and y; you may or may not be able to solve it for y, but you should simplify as much as possible!

Examples:

2. Find the general solution:

$$y' = \frac{xy^2 + 4x}{2y}$$

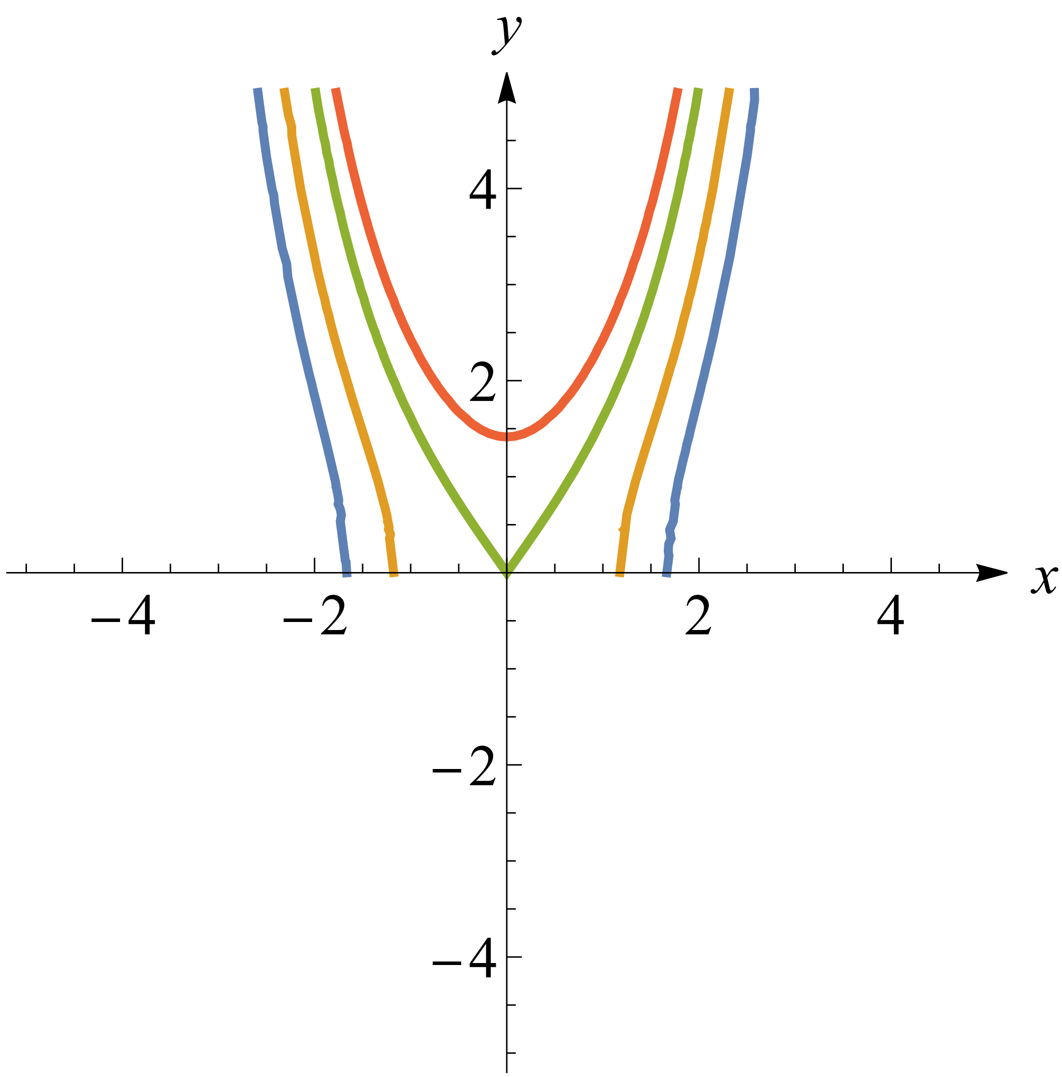
Graphs: $y^2 = Ce^{x^2/2} - 4$



Note: If you solve for y

$$y = \sqrt{Ce^{x^2/2} - 4}$$

Graphs:



3. Find the general solution:

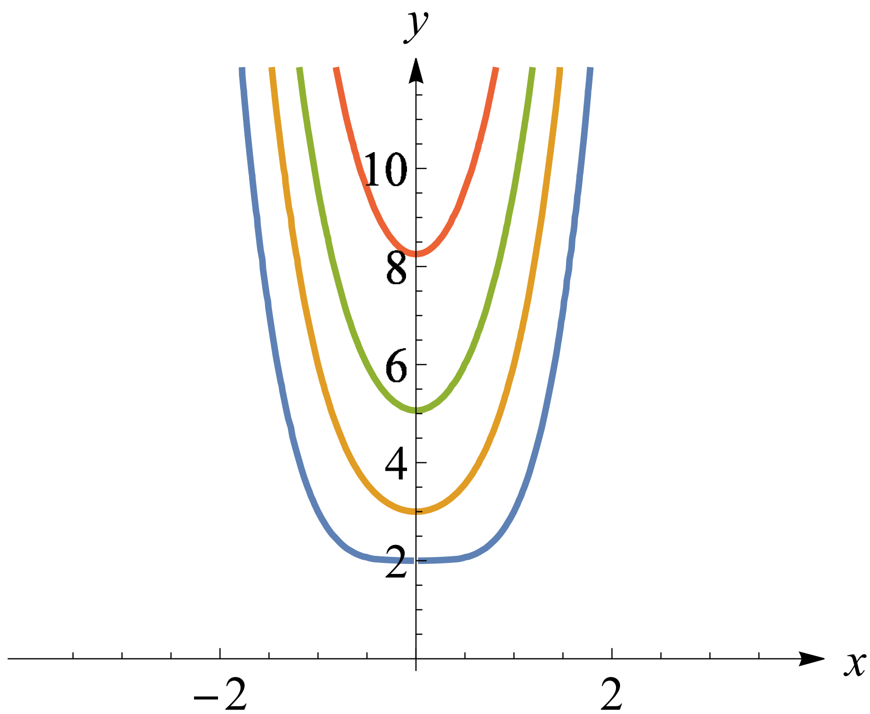
$$\frac{dy}{dx} = \frac{e^{x-y}}{1+e^x}$$

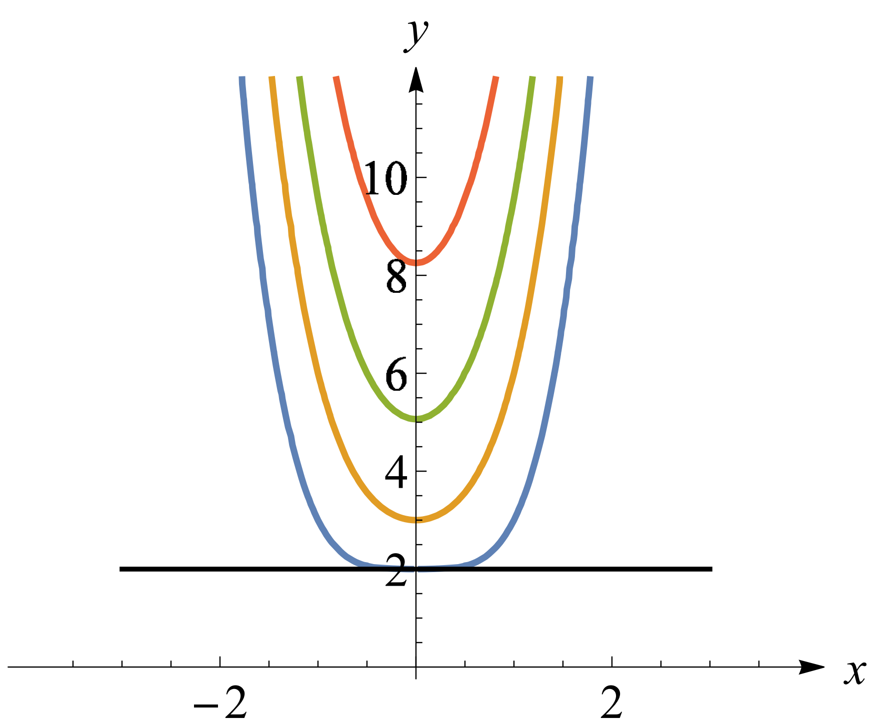
4. Find the general solution:

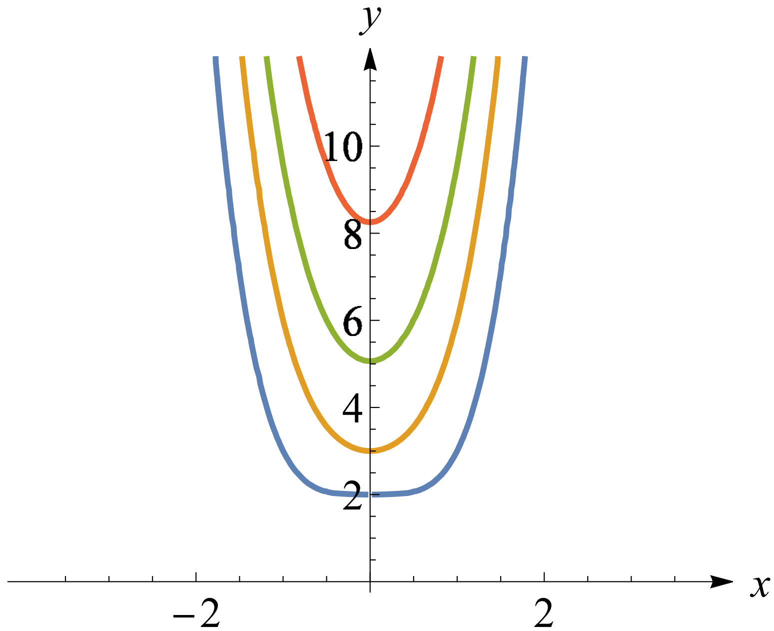
$$\frac{dy}{dx} = 4x\sqrt{y-2}$$

Singular solutions:

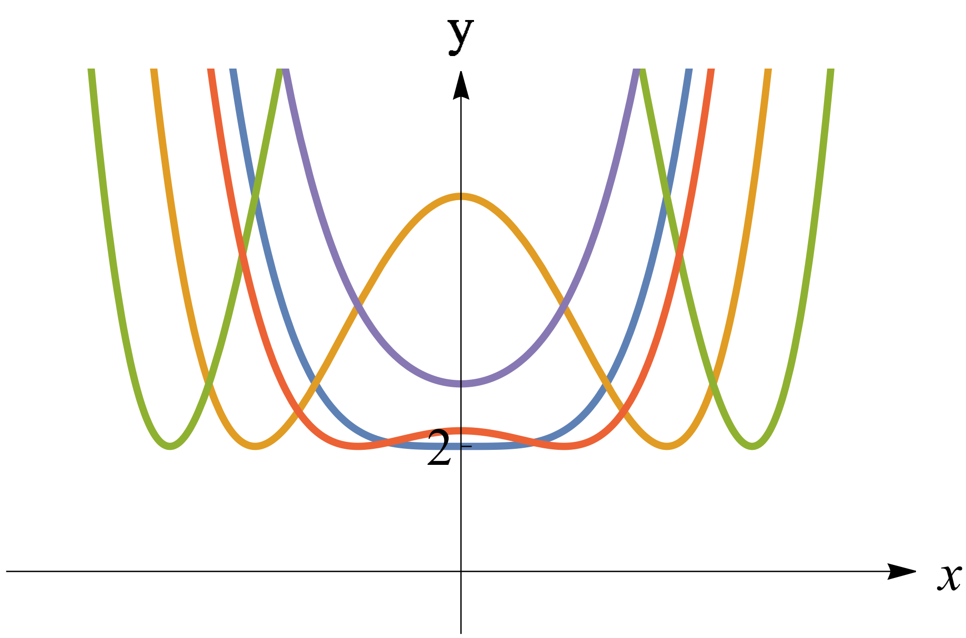
$$\sqrt{y-2} = x^2 + C$$







$$y = (x^2 + C)^2 + 2$$



5. Find the general solution:

$$\frac{dy}{dx} - xy^2 = -x$$

6. The equation

$$y' = x(y+2) \qquad \text{or} \qquad y' - xy = 2x$$

is both linear and separable. Find the general solution both ways.

Answers

1. $y' = (x-1)(y^2-1)$

2. $y^2 = Ce^{x^2/2} - 4$

3. $y = \ln\left[\ln(1+e^x) + C\right]$

4. $\sqrt{y-2} = x^2 + C$

5. $y = \dfrac{1 + Ce^{x^2}}{1 - Ce^{x^2}}$

6. $y = Ce^{x^2/2} - 2$

2.4. TRANSFORMATIONS & RELATED EQUATIONS (Text, Section 2.4)

A. Bernoulli equations

An equation that can be written in the form

$$\boxed{y' + p(x)y = q(x)y^k, \quad k \neq 0, 1}$$

is called a **Bernoulli equation**.

Note: (1) If $k = 0$ or 1, then the equation is linear.

(2) The left-hand side of a Bernoulli equation is the same as the left-hand side of a linear equation in standard form. A Bernoulli equation is ”close” to being a linear equation.

Examples: $y' + p(x)y = q(x)y^k$

1. Find the general solution of:

$$y' - 4y = 2e^x \sqrt{y}$$

Solution Method: The change of variable

$$v = y^{1-k}, \quad v' = (1-k)y^{-k}y'$$

transforms the Bernoulli equation

$$y' + p(x)y = q(x)y^k$$

into

$$v' + (1-k)p(x)v = (1-k)q(x).$$

which is:

$$v' + P(x)v = Q(x)$$

a linear equation in x and v

2. Find the general solution of:

$$xy' + y = 2x^4 y^3$$

3. Find the general solution of:

$$xy' = \frac{4e^{2x}}{x^3 y} - 2y$$

4. Find the general solution of:

$$xyy' = x^2 + 2y^2$$

Answers

1. $y^{1/2} = Ce^{2x} - e^x$

2. $$y^2 = \frac{1}{Cx^2 - 2x^4}$$

3. $$y = \frac{2e^{2x} + C}{x^4}$$

4. $y^2 = Cx^4 - x^2$

B. Homogeneous equations

$$y' = f(x, y) \tag{1}$$

is a **homogeneous equation** if

$$\boxed{f(tx, ty) = f(x, y)}$$

Solution Method: The change of dependent variable

$$y = vx, \quad y' = v + xv'$$

transforms a homogeneous equation into a separable equation:

$$y' = f(x, y) \;\to\; v + xv' = f(x, vx) = f(1, v)$$

which can be written

$$\frac{1}{f(1, v) - v}\, dv = \frac{1}{x}\, dx;$$

the variables are separated.

Examples:

1. Find the general solution:

$$y' = \frac{x^2 + y^2}{2xy}$$

2. Find the general solution:

$$\frac{dy}{dx} = \frac{x^2 e^{y/x} + y^2}{xy}$$

3. Find the general solution of:

$$xy' = y + \sqrt{x^2 - y^2}$$

4. Find the general solution of:

$xyy' = x^2 + 2y^2$ **(same as Bernoulli Pb 4)**

Answers

1. $y^2 = x^2 - Cx$

2. $y + x = e^{y/x}\,[Cx - x\ln x]$

3. $y = x\sin(\ln x + C)$

4. $y^2 = Cx^4 - x^2$

Signals: homogeneous equations:

- If the equation contains a term such as $e^{y/x}$, $\sin(y/x)$, $\cos(y/x)$, etc., then the equation is likely to be homogeneous;

- If f is an algebraic expression

$$f(x,y) = \frac{P(x,y)}{Q(x,y)},$$

and all the terms have **the same degree**, then the equation is homogeneous.