Chapter 2, Part 2

2.5. Applications (Text, Section 2.5)

Orthogonal trajectories

Exponential Growth/Decay

Newton's Law of Cooling/Heating

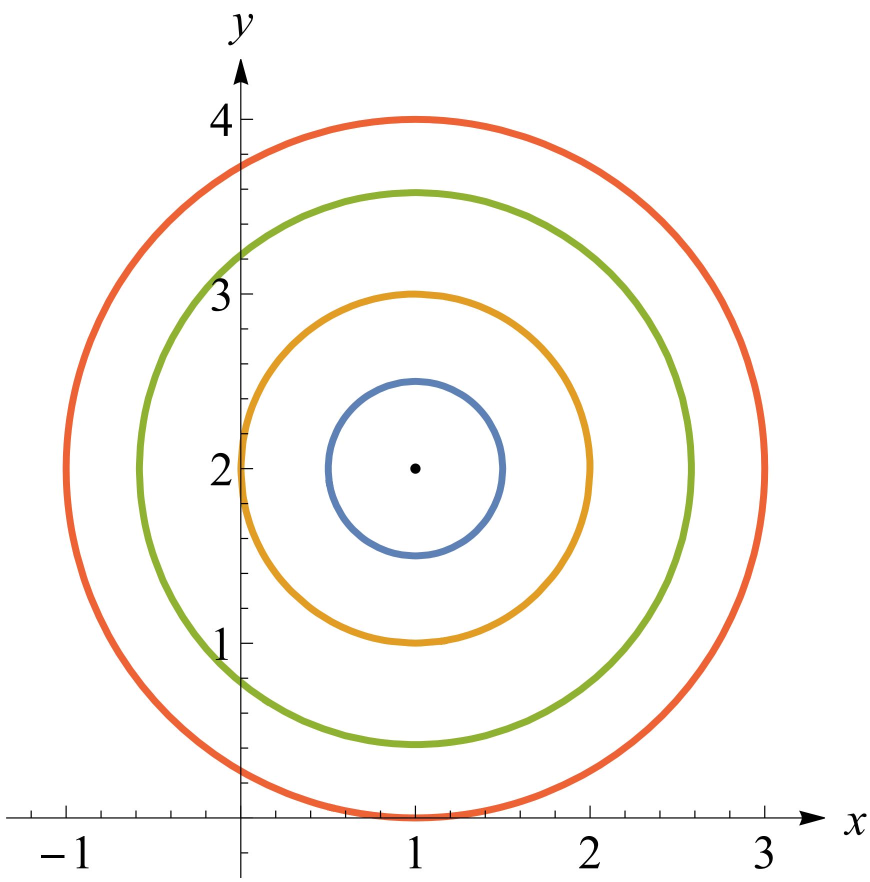
Limited Growth (Logistic Equation)

Miscellaneous Models

2.5.1. Orthogonal Trajectories

Example: Family of circles, center at $(1, 2)$:

$$(x-1)^2 + (y-2)^2 = C$$



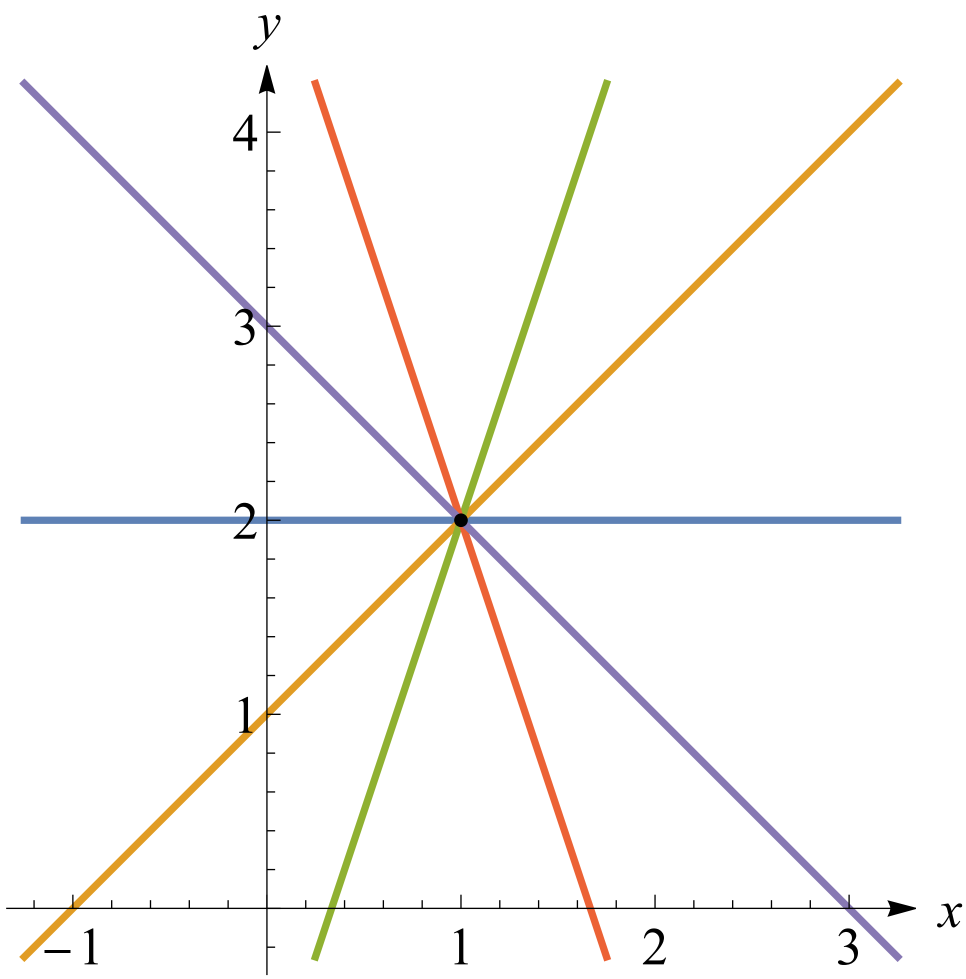
DE for the family:

$$(x-1)^2 + (y-2)^2 = C$$

$$y' = -\frac{x-1}{y-2}$$

Family of lines through $(1,2)$:

$$y-2=K(x-1)$$



DE for this family:

$$y - 2 = K(x - 1)$$

$$y' = \frac{y-2}{x-1}$$

circles: slope of tangent line at (x, y)

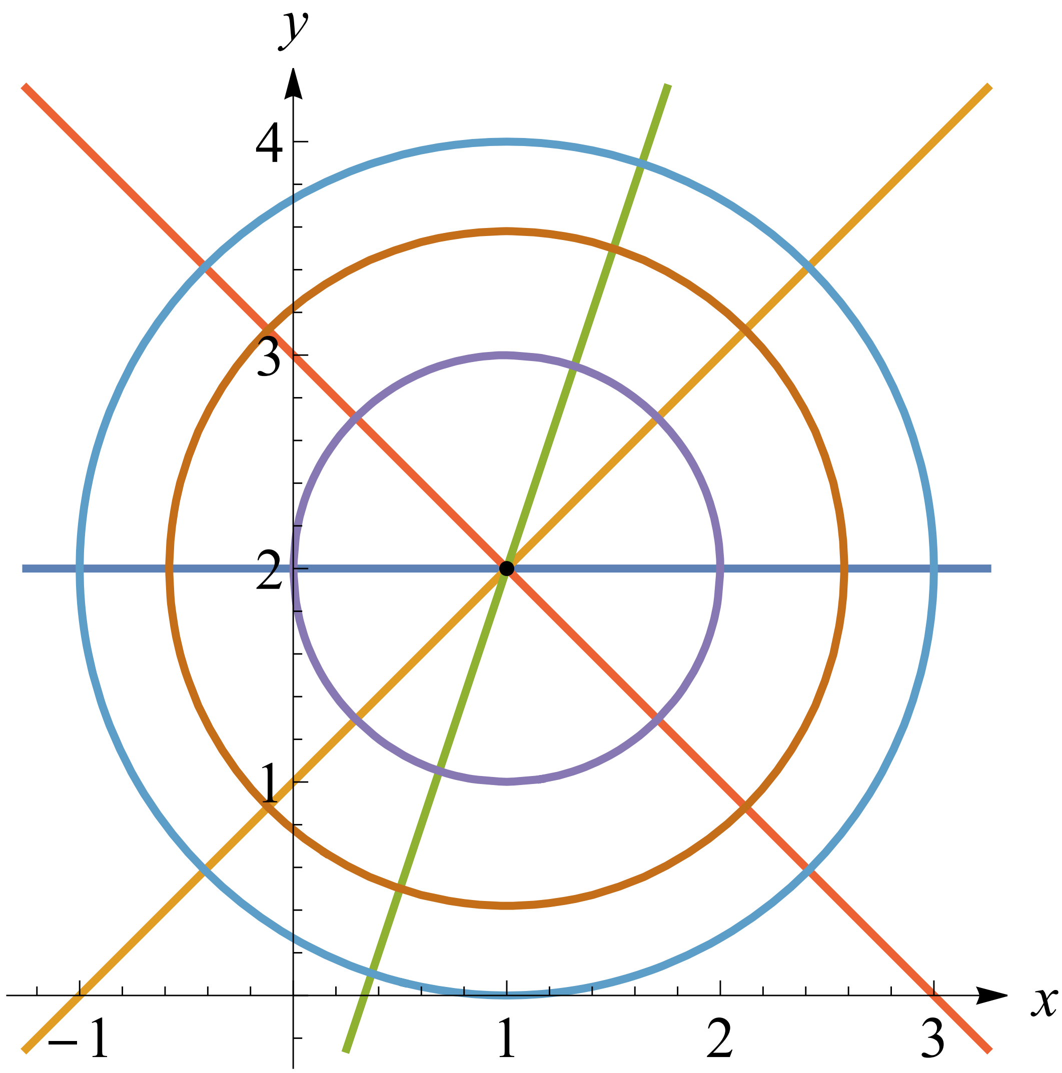
$$y' = -\frac{x-1}{y-2}$$

lines: slope of tangent line at (x, y)

$$y' = \frac{y-2}{x-1}$$

Negative reciprocals!! The lines and circles are perpendicular (orthogonal) to each other.

The lines and the circles:



Given a one-parameter family of curves

$$F(x, y, C) = 0.$$

A curve that intersects each member of the family at right angles (orthogonally) is called an **orthogonal trajectory** of the family.

If

$$F(x, y, C) = 0 \quad \text{and} \quad G(x, y, K) = 0$$

are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be **orthogonal trajectories**.

A **procedure** for finding a family of orthogonal trajectories

$$G(x, y, K) = 0$$

for a given family of curves

$$F(x, y, C) = 0$$

Step 1. Determine the differential equation for the given family (**recall Chapter 1 problems**)

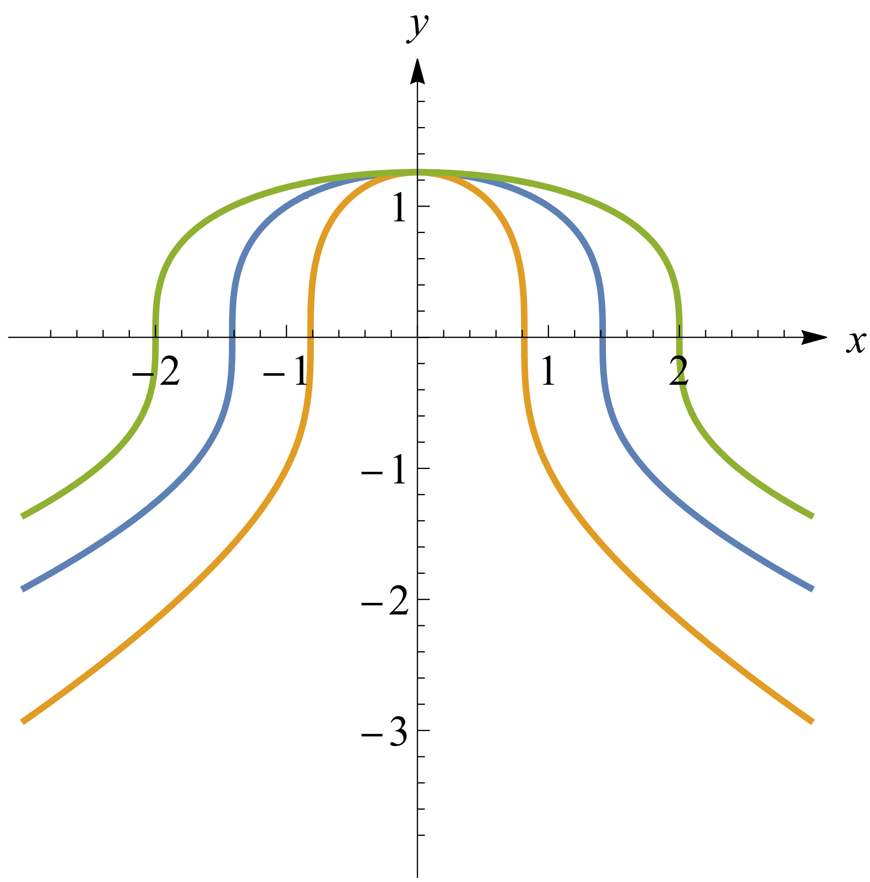
$$F(x, y, C) = 0.$$

Step 2. Replace y' in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

Step 3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Examples

1. Find the family of orthogonal trajectories of $y^3 = Cx^2 + 2$

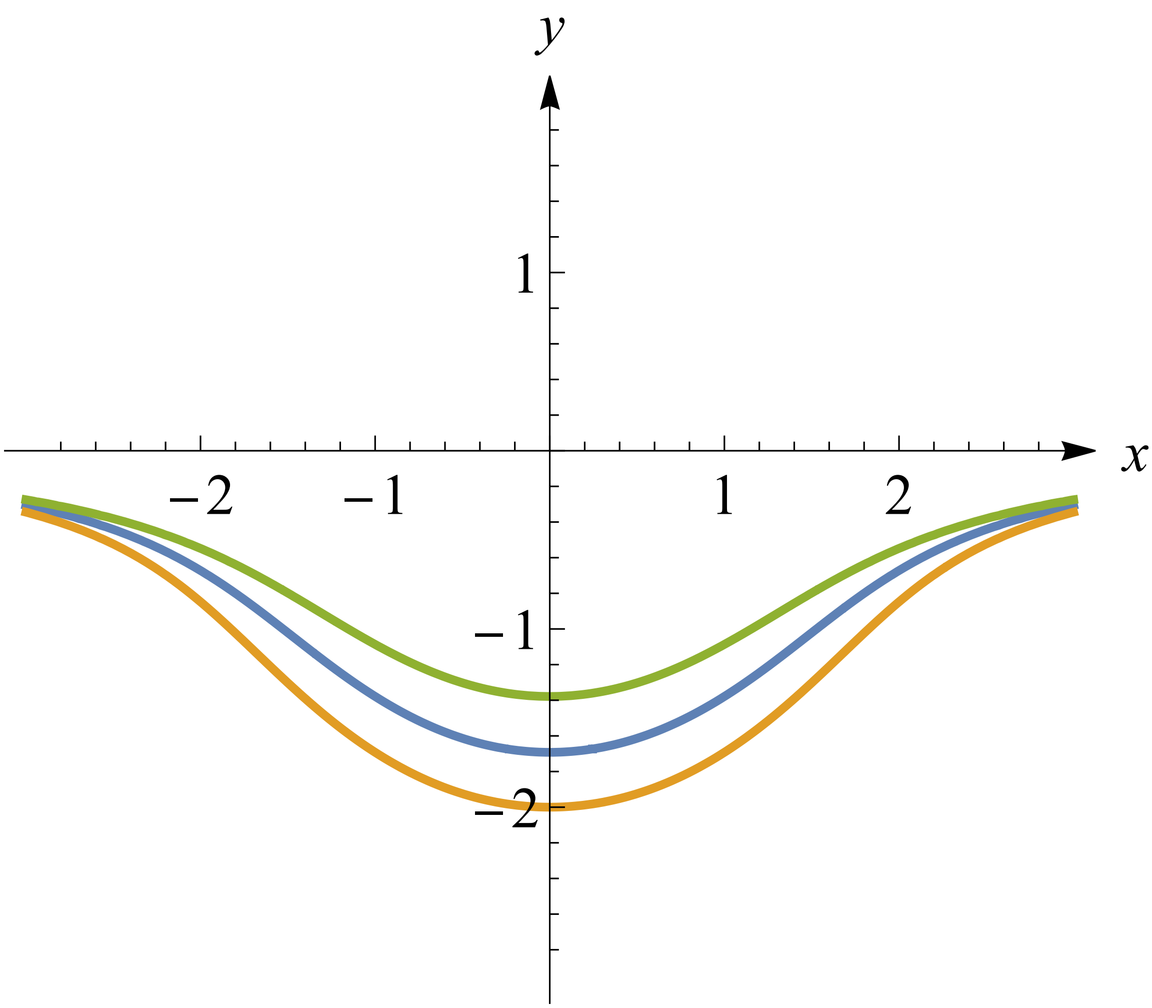


$y^3 = Cx^2 + 2, \ C = -1/2, \ -1, -3$

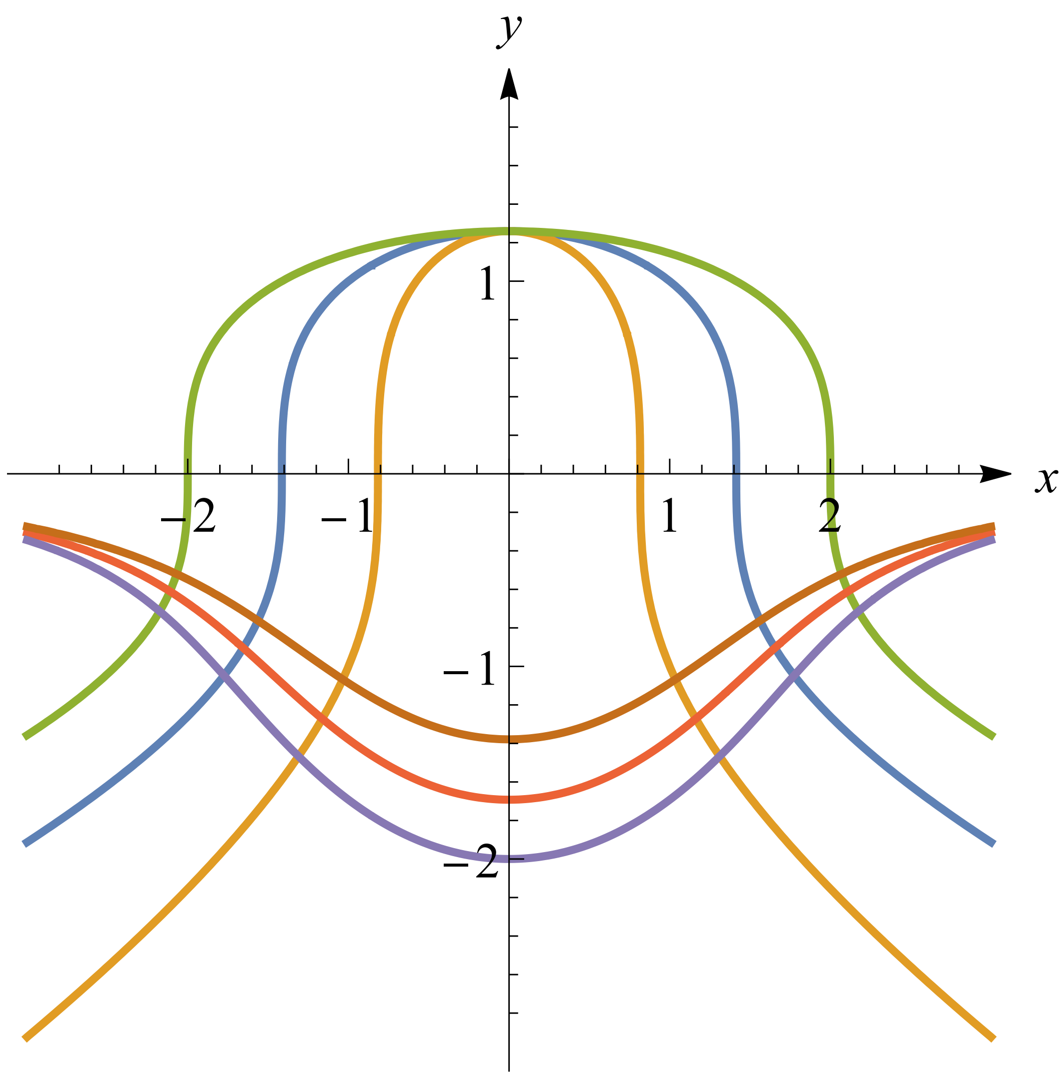
$$y^3 = Cx^2 + 2$$

Orthogonal trajectories:

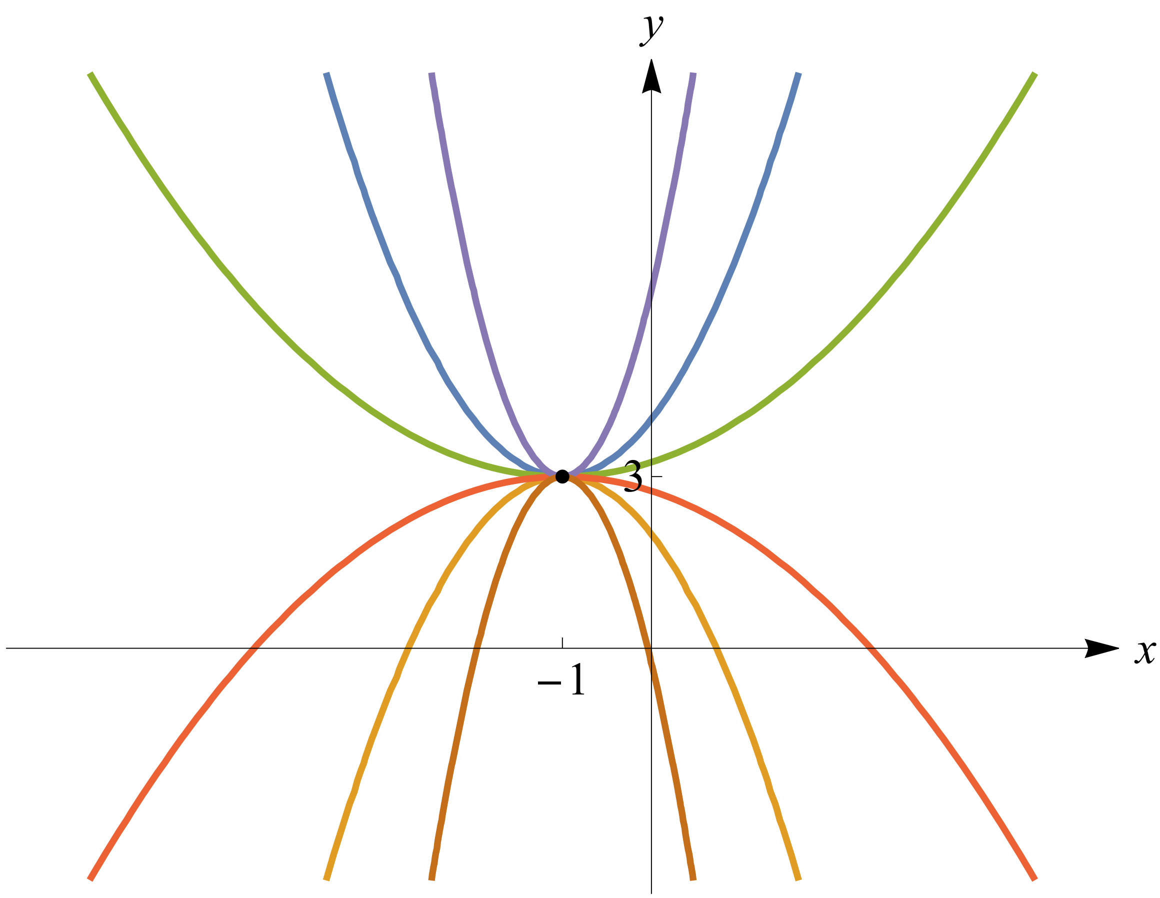
$$3x^2 + 2y^2 + \frac{8}{y} = C$$



Graphed together:



2. Find the orthogonal trajectories of the family of parabolas with vertical axis and vertex at the point $(-1, 3)$.

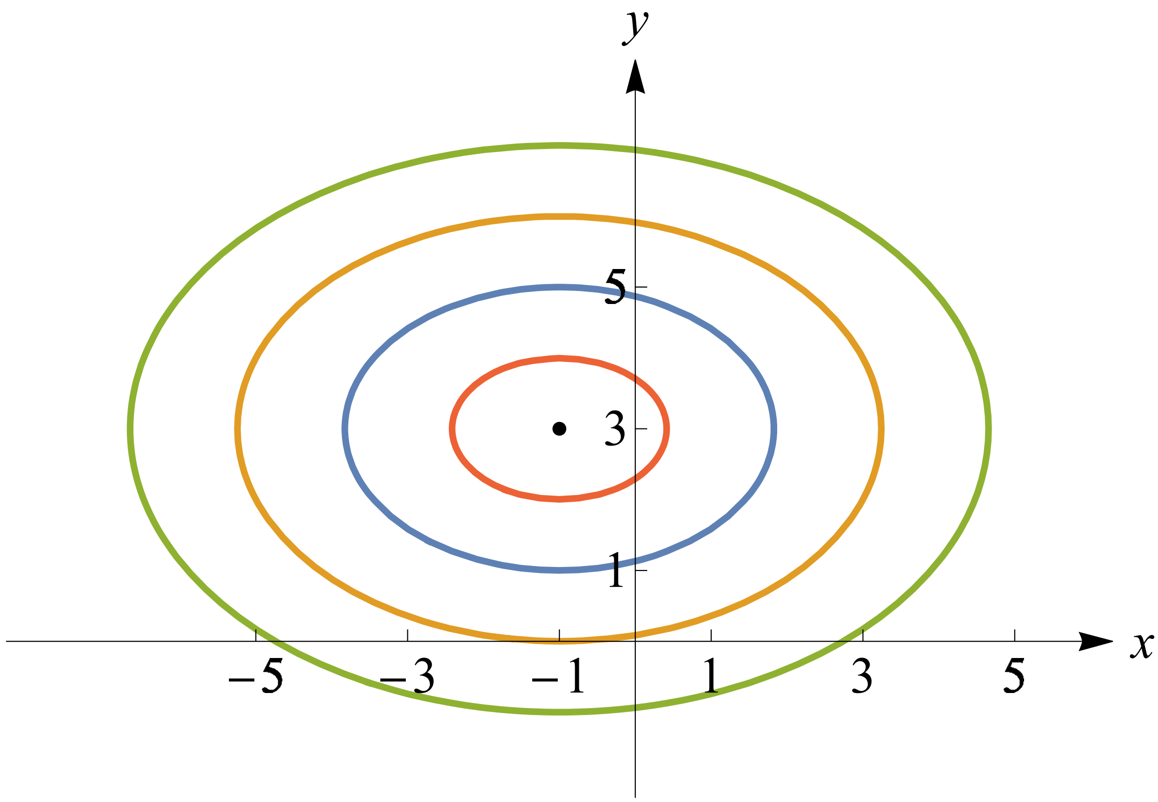


Differential equation for the family:

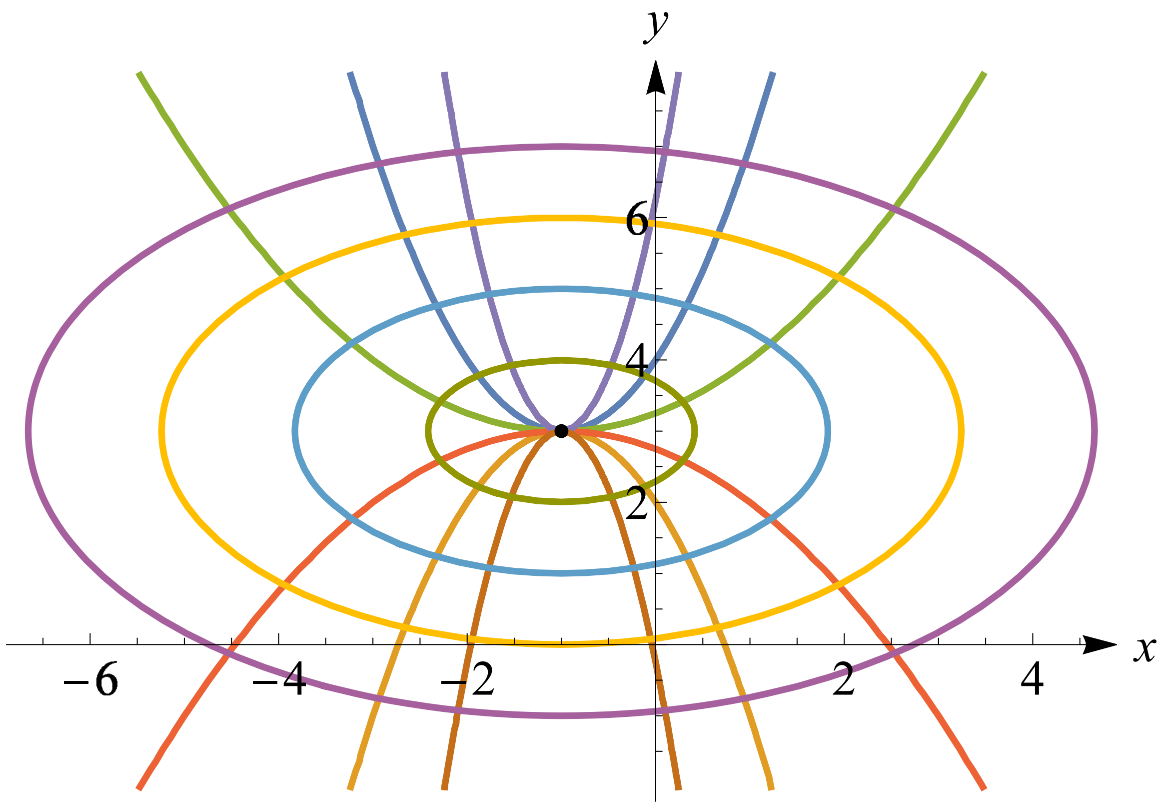
Orthogonal trajectories:

$$\frac{1}{2}(x+1)^2 + (y-3)^2 = C$$

$$\frac{1}{2}(x+1)^2 + (y-3)^2 = C \quad \text{(ellipses)}$$



Both families:



2.5.2. Radioactive Decay/Exponential Growth

Radioactive Decay

"Experiment:" The rate of decay of a radioactive material at time t is proportional to the amount of material present at time t.

Let $A = A(t)$ be the amount of radioactive material present at time t.

Mathematical Model

$$\frac{dA}{dt} = k\,A, \quad k \text{ a constant,}$$

Note: $k < 0$

$$A(0) = A_0, \quad \text{the initial amount.}$$

Solution: $A(t) = A_0 e^{kt}$.

Half-life: The length of time required for the material to decay to one-half the original amount.

$$T = \frac{\ln 1/2}{k} = \frac{-\ln 2}{k}.$$

Note: The decay model is often written equivalently as:

$$\frac{dA}{dt} = -r\,A, \quad r > 0 \quad \text{constant},$$

$$A(0) = A_0, \quad \text{the initial amount.}$$

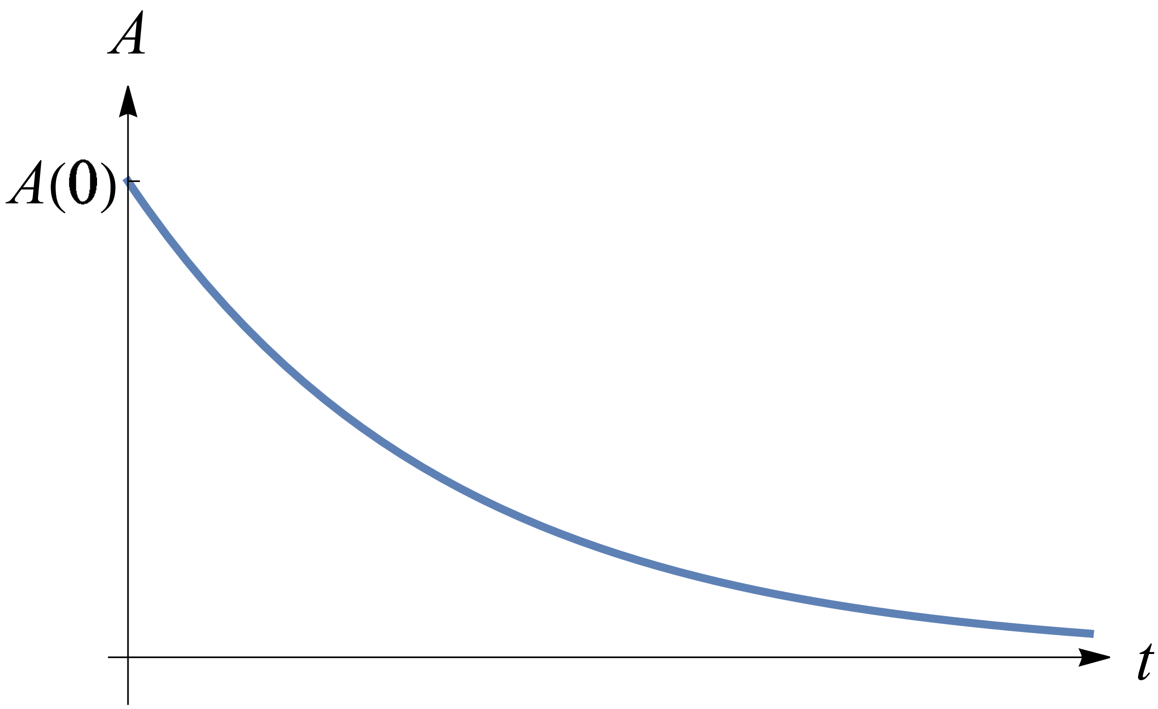
Solution: $A(t) = A_0 e^{-rt}$

r is the **decay rate.**

Solution: $A(t) = A_0\, e^{-rt}$.

Half-life: $T = \dfrac{\ln 2}{r}$.

Graph:



Note: $\lim_{t \to \infty} A(t) = 0$

Example: A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

1. An expression for the mass of the material remaining at any time t.

2. The mass of the material after 4 hours.

3. How long will it take for 75% of the material to decay?

$t \approx 26.32$ hours

4. The half-life of the material.

$$T = \frac{-\ln 2}{k} = \frac{-2\ln 2}{\ln(9/10)} \approx 13.16 \text{ hours}$$

Same problem using

$$\frac{dA}{dt} = -rA, \;\; A(0) = 50$$

Exponential Growth

"Experiment:" Under "ideal" conditions, the rate of increase of a population at time t is proportional to the size of the population at time t. Let $P = P(t)$ be the size of the population at time t.

Mathematical Model

$$\frac{dP}{dt} = k\,P, \qquad k > 0 \quad \text{constant.}$$

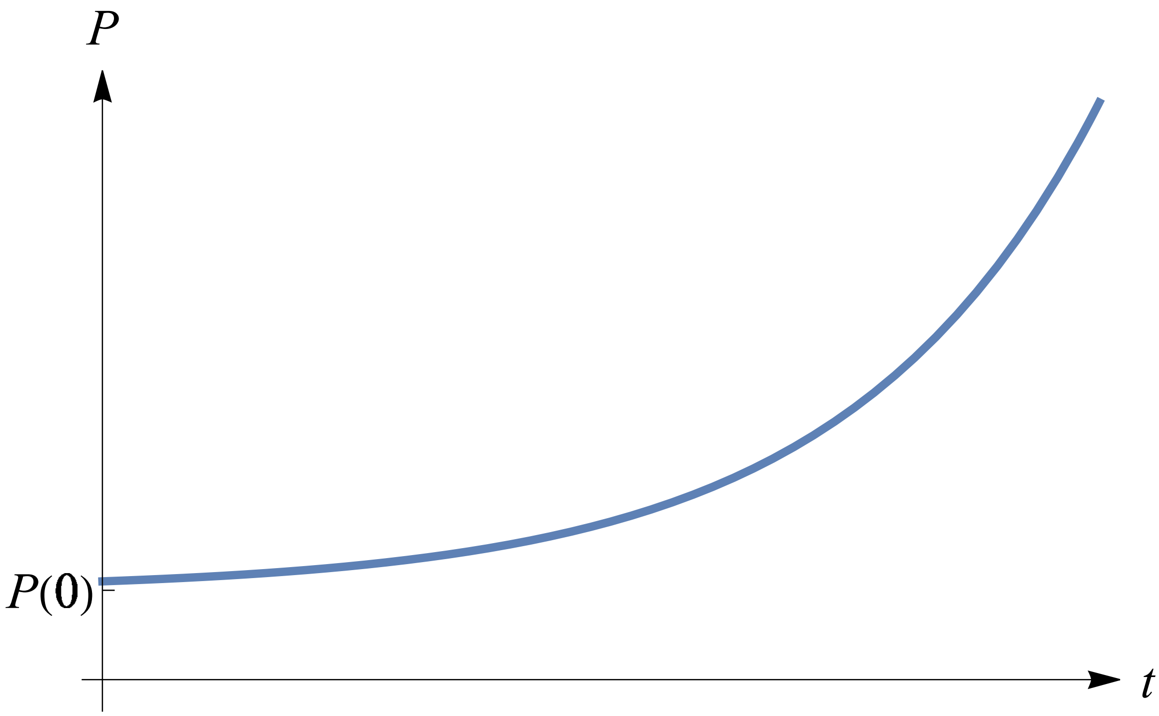
$$P(0) = P_0, \quad \text{the initial population.}$$

k is the **growth rate.**

Solution: $\quad P(t) = P_0\, e^{kt}.$

Doubling time: $\quad T = \dfrac{\ln 2}{k}.$

Graph:



Note: $\lim_{t \to \infty} = \infty$

Example: Scientists observed that a small colony of penguins on a remote Antarctic island obeys the population growth law. There were 1000 penguins initially and 1500 penguins 12 months later.

Here are the penguins (except for that one guy)

(a) Find the growth constant and give the penguin population at any time t.

Answer: $P(t) = 1000\left(\frac{3}{2}\right)^{t/12}$

(b) What is the penguin population after 3 years?

(c) How long will it take for the penguin population to double in size?

Answer: $T = \dfrac{\ln 2}{k} = \dfrac{12 \ln 2}{\ln(3/2)} \approx$ 20.5 mos

(d) How long will it take for the penguin population to reach 10,000 penguins?

Answer: $t = \dfrac{12\ln(10)}{\ln(3/2)} \approx 68$ mos, 5.7 yrs.

Example: In 2000 the world population was approximately 6.1 billion and in the year 2010 it was approximately 7.0 billion. Assume that the population increases at a rate proportional to the size of population.

(a) Find the growth constant and give the world population at any time t.

Answer: $P(t) = 6.1\left(\frac{7.0}{6.1}\right)^{t/10}$

(b) How long will it take for the world population to reach 12.2 billion (double the 2000 population)?

Answer: $T \approx 50.4$ years (doubling time), (12.2 billion on 6/24/2050)

(c) The world population on 1/1/2023 is reported to be about 8.02 billion. What population does the formula in (1) predict for the year 2024

Answer: $P(23) \approx 8.9$ billion

Example: It is estimated that the arable land on earth can support a maximum of 30 billion people. Extrapolate from the data given in the previous example to estimate the year when the food supply becomes insufficient to support the world population.

Solve $6.1\left(\frac{7}{6.1}\right)^{t/10} = 30$ for t

$t \approx 116$ year 2116

Example: Business/Economics:

Money deposited in a bank that pays $r\%$ interest compounded continuously obeys the population growth law.

Suppose that \$1000 is deposited in the bank at 6% interest. How long will it take for the money to double?

Answer: $T = \dfrac{\ln 2}{0.06} \approx \dfrac{0.69}{.06} = \dfrac{69}{6} =$ 11.5 years

In general, the doubling time for an investment P at an interest rate r% is:

$$T = \frac{69}{r}$$

BUT, people in the financial world use the **Rule of 72:**

The doubling time for an investment at an interest rate of r% is:

$$T = \frac{72}{r}$$

2.5.3. Newton's Law of Cooling

"Experiment:" The rate of change of the temperature of an object at time t is proportional to the difference between the temperature of the object $u = u(t)$ and the (constant) temperature σ of the surrounding medium (e.g., air or water)

$$\frac{du}{dt} = k(u - \sigma)$$

Note: $k < 0$

Mathematical Model

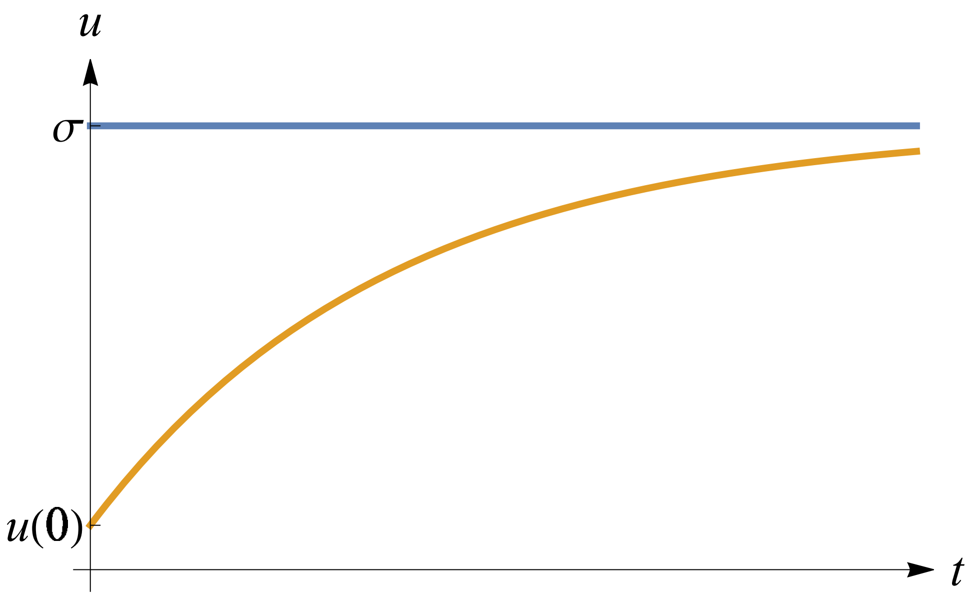
$$\frac{du}{dt} = -k(u - \sigma), \ \ k > 0 \text{ constant},$$

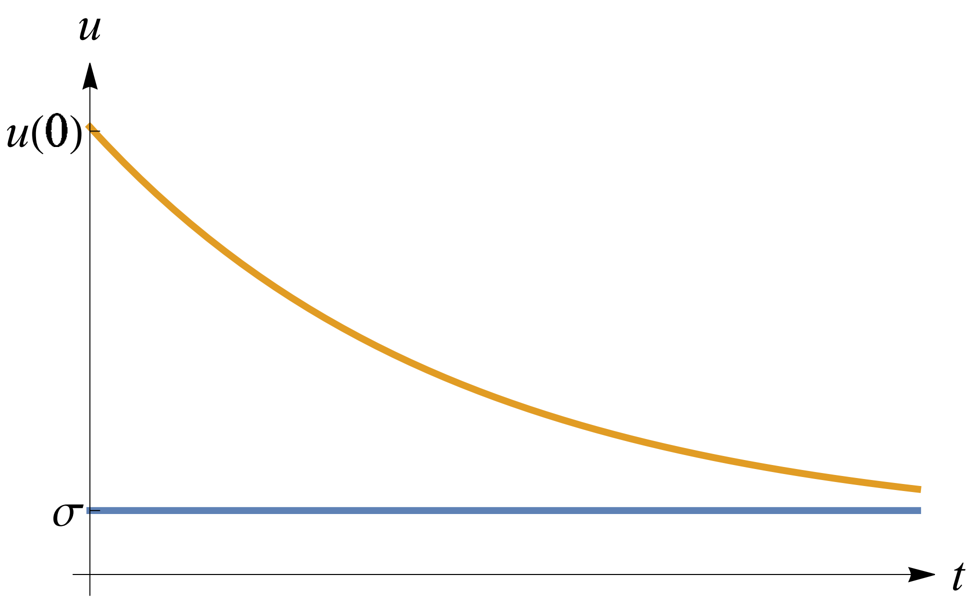
$u(0) = u_0$, the initial temperature.

Solution:

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}$$

Graphs:





Note: $\lim_{t\to\infty} u(t) = \sigma$

Example: A corpse is discovered at 10 p.m. and its temperature is determined to be $85^o\,F$. Two hours later, its temperature is $74^o\,F$. If the ambient temperature is $68^o\,F$, estimate the time of death.

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}$$

$$= 68 + (85 - 68)e^{-kt} = 68 + 17e^{-kt}$$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$$u(t) = 68 + 17e^{-kt}$$

2.5.6. “Limited” Growth – the Logistic Equation

“Experiment:” Given a population of size M. The spread of an infectious disease at time t (or information, or ...) is proportional to the product of the number of people who have the disease $P(t)$ and the number of people who do not $M - P(t)$.

Mathematical Model:

$$\frac{dP}{dt} = kP(M - P), \; k > 0 \text{ constant,}$$

$$= kMP - kP^2$$

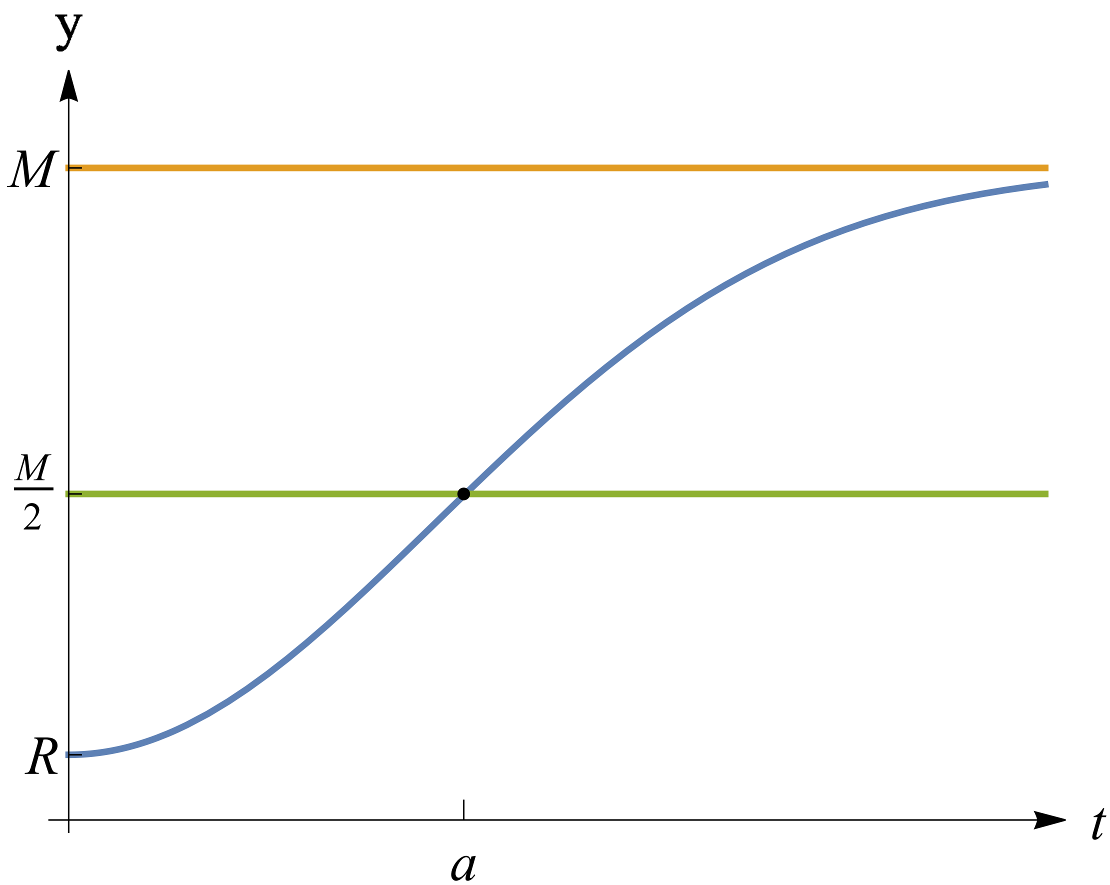
$P(0) = R$ (the number of people who have the disease initially)

Solution: The differential equation is both separable and Bernoulli.

Solution:

$$P(t) = \frac{MR}{R + (M - R)e^{-Mkt}}$$

Graph:



2.5.7 Mathematical Modeling

1. A disease is spreading through a small cruise ship with 200 passengers and crew. Let $P(t)$ be the number of people who have the disease at time t. Suppose that 15 people had the disease initially and that the rate at which the disease is spreading at time t is proportional to the number of people who don't have the disease.

a. Give the mathematical model (initial-value problem) which describes the spread of the disease.

b. Find the solution.

$$\frac{dP}{dt} = k(200 - P), \;\; P(0) = 15$$

$$P(t) = 200 - 185e^{-kt}.$$

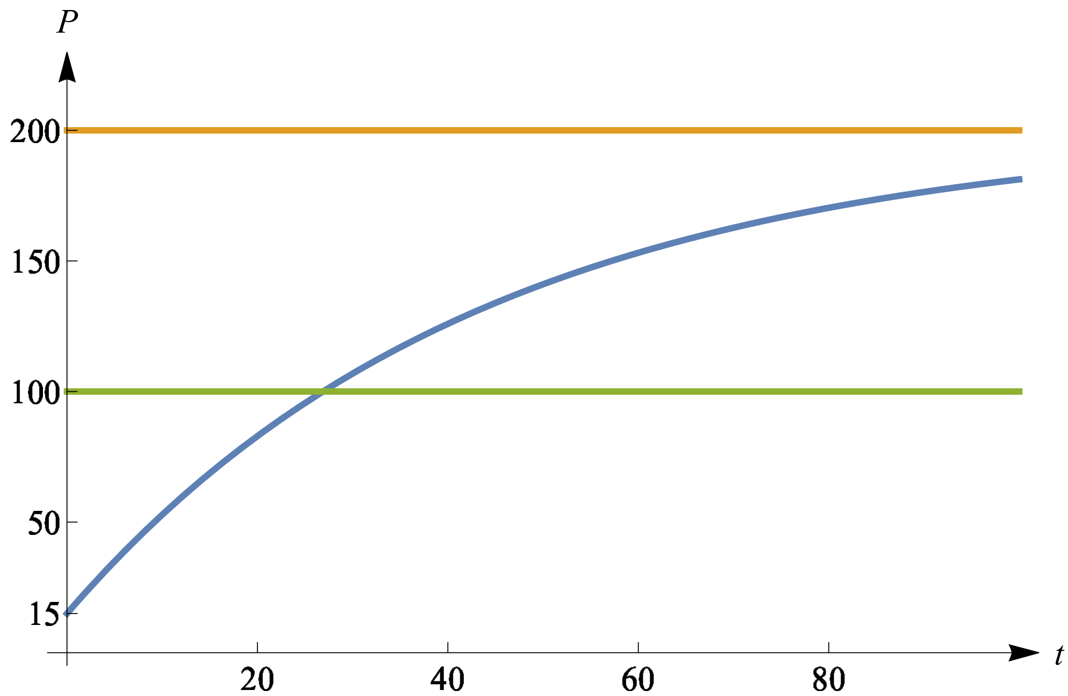
c. Suppose that 35 people are sick after 5 days. (a) How many people will be sick after t days? (b) After 15 days? (c) How long will it take for half the people to be sick?

(a) $P(t) = 200 - 185\left(\frac{33}{37}\right)^{t/5}.$

(b) $P(15) =$

(c) $P(t) = 100 \quad t \approx 27$

Graph:



d. Find $\lim_{t\to\infty} P(t)$ and interpret the result. $P(t) = 200 - 185\left(\frac{33}{37}\right)^{t/5}$.

$\lim_{t\to\infty} P(t) = 200$; everyone gets sick.

2. A 1000-gallon cylindrical tank, initially full of water, develops a leak at the bottom. Suppose that the water drains off a rate proportional to the product of the time elapsed and the amount of water present. Let $A(t)$ be the amount of water in the tank at time t.

a. Give the mathematical model (initial-value problem) which describes the process.

b. Find the solution.

$$\frac{dA}{dt} = ktA, \; k < 0, \; A(0) = 1000$$

$$A(t) = 1000e^{kt^2/2}.$$

c. Given that 200 gallons of water leak out in the first 10 minutes, find the amount of water, $A(t)$, left in the tank t minutes after the leak develops.

$$A(t) = 1000\left(\frac{4}{5}\right)^{t^2/100}.$$

3. A 1000-gallon tank, initially containing 900 gallons of water, develops a leak at the bottom. Suppose that the water drains off a rate proportional to the square root of the amount of water present. Let $A(t)$ be the amount of water in the tank at time t.

a. Give the mathematical model (initial-value problem) which describes the process.

b. Find the solution

$$\frac{dA}{dt} = k\sqrt{A},\ k < 0,\ A(0) = 900$$

$A(t) = \left(\frac{1}{2}kt + 30\right)^2.$

4. A disease is spreading through a small cruise ship with 200 passengers and crew. Let $P(t)$ be the number of people who have the disease at time t. Suppose that 15 people had the disease initially and that the rate at which the disease is spreading at time t is proportional to the product of the time elapsed and the number of people who don't have the disease.

a. Give the mathematical model (initial-value problem) which describes the process.

b. Find the solution.

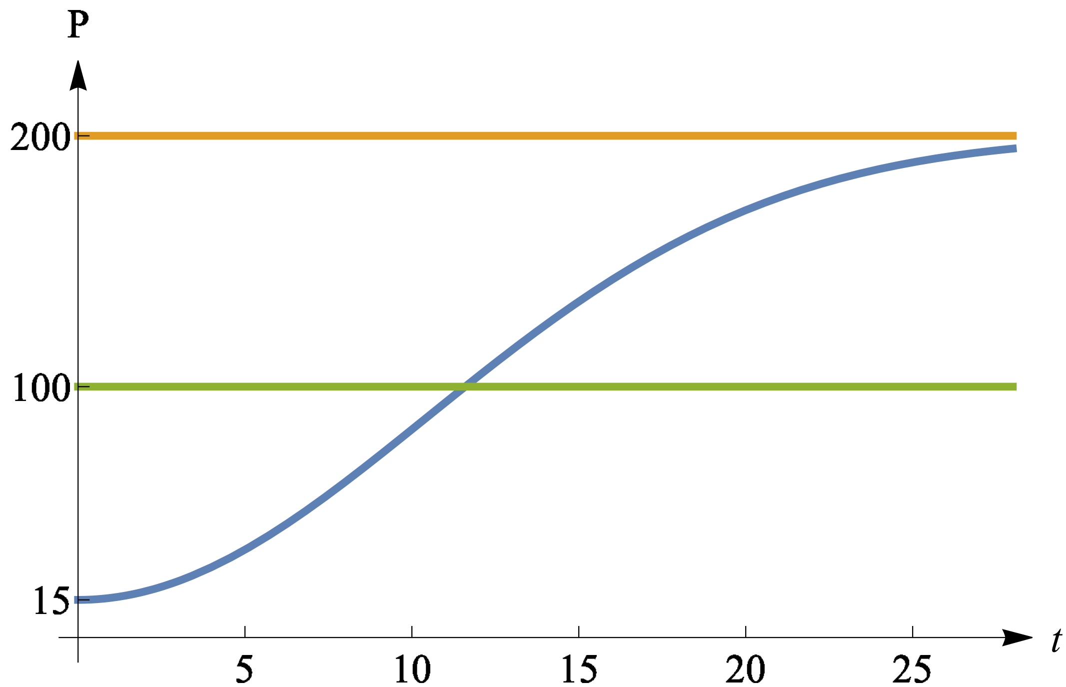
$$\frac{dP}{dt} = kt(200 - P), \; P(0) = 15$$

$$P(t) = 200 - 185e^{-kt^2/2}.$$

c. Suppose that 35 people are sick after 5 days. How many people will be sick after t days?

$$P(t) = 200 - 185\left(\frac{33}{37}\right)^{t^2/25}.$$

Graph:



Existence and Uniqueness Theorem: Given the initial-value problem: $y' = f(x, y)$ $y(a) = b$.

If f and $\partial f/\partial y$ are continuous on a rectangle

$$R: \; a \le x \le b, \; c \le y \le b,$$

then there is an interval

$$a - h \le x \le a + h$$

on which the initial-value problem has a unique solution $y = y(x)$.