Second Order Linear Differential Equations (Text: Section 3.1)

A **second order linear differential equation** is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where p, q, and f are continuous functions on some interval I.

The functions p and q are called the **coefficients** of the equation.

In applications, the function f is often called the **forcing function** (see Section 3.6).

“Linear”

Set $L[y] = y'' + p(x)y' + q(x)y$. Then, for any two twice differentiable functions $y_1(x)$ and $y_2(x)$,

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$$

and, for any constant c,

$$L[cy(x)] = cL[y(x)].$$

That is, L is a **linear differential operator**.

$$L[y] = y'' + py' + qy$$

$$L[y_1 + y_2] =$$

$$(y_1 + y_2)'' + p\,(y_1 + y_2)' + q\,(y_1 + y_2)$$

$$= y_1'' + y_2'' + p\left(y_1' + y_2'\right) + q\,(y_1 + y_2)$$

$$= y_1'' + y_2'' + py_1' + py_2' + qy_1 + qy_2$$

$$= \left(y_1'' + py_1' + qy_1\right) + \left(y_2'' + py_2' + qy_2\right)$$

$$= L[y_1] + L[y_2]$$

$$L[cy] = (cy)'' + p(cy)' + q(cy)$$

$$= cy'' + pcy' + qcy = c\left(y'' + py' + qy\right)$$

$$= cL[y]$$

Existence and Uniqueness

THEOREM: Given the second order linear equation (1). Let a be any point on the interval I, and let α and β be any two real numbers. Then the initial-value problem

$$y'' + p(x)\,y' + q(x)\,y = f(x),$$

$$y(a) = \alpha,\ \ y'(a) = \beta$$

has a unique solution.

Homogeneous/Nonhomogeneous Equations

The linear differential equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (1)$$

is **homogeneous*** if the function f on the right side is 0 for all $x \in I$. That is,

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

is a **linear homogeneous** equation.

*There is no relation between the term here and 1st order "homogeneous" equations in 2.4

If f is not the zero function on I,

that is, if $f(x) \neq 0$ for some $x \in I$,

then

$$y'' + p(x)y' + q(x)y = f(x)$$

is a **linear nonhomogeneous** equation.

Homogeneous Equations (Text, Section 3.2)

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad \text{(H)}$$

where p and q are continuous functions on some interval I.

The zero function, $y(x) = 0$ for all $x \in I$, ($y \equiv 0$) is a solution of (H):

The zero solution $y \equiv 0$ is called the **trivial solution**. Any other solution is a **nontrivial** solution.

Recall, Example 8, Chap. 1, pg 20:

Find a value of r, if possible, such that $y = x^r$ is a solution of

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0.$$

Solutions:

$y \equiv 0$ is a solution (trivial)

$y_1 = x^{-1}, \quad y_2 = x^3$ are solutions

$y = C_1x^{-1} + C_2x^3$ for any constants C_1, C_2

Basic Theorems

THEOREM 1: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then

$$u(x) = y_1(x) + y_2(x)$$

is also a solution of (H).

The sum of any two solutions of (H) is also a solution of (H). (Some call this property the *superposition principle*).

Proof:

y_1 and y_2 are solutions. Therefore,

$$L[y_1] = 0 \quad \text{and} \quad L[y_2] = 0$$

L is linear. Now

THEOREM 2: If $y = y(x)$ is a solution of (H) and if C is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

Proof: y is a solution means $L[y] = 0$.

L is linear:

Any constant multiple of a solution of (H) is also a solution of (H).

DEFINITION: Let $y = y_1(x)$ and $y = y_2(x)$ be functions defined on some interval I, and let C_1 and C_2 be real numbers. The expression

$$C_1 y_1(x) + C_2 y_2(x)$$

is called a **linear combination** of y_1 and y_2.

Theorems 1 & 2 can be restated as:

THEOREM 3: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), and if C_1 and C_2 are any two real numbers, then

$$y(x) = C_1\, y_1(x) + C_2\, y_2(x)$$

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).

NOTE: $y(x) = C_1\, y_1(x) + C_2\, y_2(x)$

is a two-parameter family which ”looks like“ the general solution.

Is it???

Some Examples from Chapter 1:

1.

$$y_1 = \cos 3x \text{ and } y_2 = \sin 3x$$

are solutions of

$$y'' + 9y = 0 \quad \text{(Chap 1, p. 46)}$$

$$y = C_1 \cos 3x + C_2 \sin 3x$$

is the general solution.

2. $y_1 = e^{-2x}$ and $y_2 = e^{4x}$

are solutions of

$$y'' - 2y' - 8y = 0 \quad \text{(Chap 1, p. 54)}$$

and

$$y = C_1 e^{-2x} + C_2 e^{4x}$$

is the general solution.

3. $y_1 = x$ and $y_2 = x^3$

are solutions of

$$y'' - \frac{3}{x}y' - \frac{3}{x^2}y = 0 \quad \text{(Chap 1, p. 55)}$$

and

$$y = C_1 x + C_2 x^3$$

is the general solution.

Example: $y'' - \frac{1}{x}y' - \frac{15}{x^2}y = 0$

a. Solutions

$$y_1(x) = x^5, \quad y_2(x) = 3x^5$$

General solution:

$$y = C_1 x^5 + C_2(3x^5) \quad ??$$

That is, is EVERY solution a linear combination of

$$y_1 \quad \text{and} \quad y_2\ ?$$

ANSWER: **NO !!!**

$y = x^{-3}$ is a solution: (verify this)

AND

$$x^{-3} \neq C_1 x^5 + C_2(3x^5)$$

since

$$C_1 x^5 + C_2(3x^5) = (C_1 + 3C_2)x^5 = M\, x^5$$

CLEARLY, x^{-3} is **NOT** a constant multiple of x^5.

Now consider

$$y_1(x) = x^5, \; y_2(x) = x^{-3}$$

General solution: $y = C_1 x^5 + C_2 x^{-3}$?

That is, is EVERY solution a linear combination of y_1 and y_2??

Let $y = y(x)$ be **any** solution of the equation. Suppose

$$y(1) = \qquad\qquad y'(1) =$$

$$y = C_1 x^5 + C_2 x^{-3}$$

$$y' = 5C_1 x^4 - 3C_2 x^{-4}$$

At $x = 1$:

$$C_1 + C_2 =$$

$$5C_1 - 3C_2 =$$

In general: Let

$$y = C_1 y_1(x) + C_2 y_2(x)$$

be a two-parameter family of solutions of (H). When is this the general solution of (H)?

EASY ANSWER: When y_1 and y_2 **ARE NOT CONSTANT MULTIPLES OF EACH OTHER.**

That is, y_1 and y_2 are independent of each other.

Let

$$y = C_1 y_1(x) + C_2 y_2(x)$$

be a two parameter family of solutions of (H). Choose any number $a \in I$ and let u be <u>any</u> solution of (H).

Suppose $u(a) = \alpha, \quad u'(a) = \beta$

Does the system of equations

$$C_1y_1(a) + C_2y_2(a) = \alpha$$

$$C_1y_1'(a) + C_2y_2'(a) = \beta$$

have a unique solution??

DEFINITION: Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H). The function W defined by

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the **Wronskian** of y_1, y_2.

Determinant notation:

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Examples

1. Let $y_1 = x^{-3}$, $y_2 = x^5$. Then

$$W(y_1, y_2) = \begin{vmatrix} x^{-3} & x^5 \\ -3x^{-4} & 5x^4 \end{vmatrix}$$

$$= x^{-3}(5x^4) - x^5(-3x^{-4})$$

$$= 8x$$

2. Let $y_1 = x^5$, $y_2 = 3x^5$. Then

$$W(y_1, y_2) = \begin{vmatrix} x^5 & 3x^5 \\ 5x^4 & 15x^4 \end{vmatrix}$$

$$= 15x^9 - 15x^9$$

$$= 0$$

Given two functions y_1 and y_2.

Suppose $W(y_1, y_2) \equiv 0$

Set $u(x) = \dfrac{y_2(x)}{y_1(x)}$. Then

THEOREM 4: Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ $(W(x) \equiv 0$ on $I)$; that is, y_1 and y_2 are constant multiples of each other, AND

$$y = C_1 y_1(x) + C_2 y_2(x)$$

IS NOT the general solution of (H)

OR

(ii) $W(x) \neq 0$ for all $x \in I$, AND

$$y = C_1 y_1(x) + C_2 y_2(x)$$

IS the general solution of (H)

Fundamental Set; Solution Basis; Linearly Independent Set

DEFINITION: A pair of solutions

$$y = y_1(x), \quad y = y_2(x)$$

of equation (H) forms a **fundamental set of solutions** (also called a **solution basis** or a **linearly independent set)** if y_1 and y_2 **are not constant multiples of each other**. Equivalently,

$$W[y_1, y_2](x) \neq 0 \quad \text{for all } x \in I.$$

Homogeneous Equations with Constant Coefficients (Text, Section 3.3)

Fact: In contrast to first order linear equations, there are **no general methods** for solving

$$y'' + p(x)y' + q(x)y = 0. \qquad \text{(H)}$$

But, there is a special case of (H) for which there is a solution method, namely when the equation has constant coefficients:

$$y'' + ay' + by = 0 \qquad (1)$$

where a and b are constants.

Solutions: (1) has solutions of the form

$$y = e^{rx} \quad \text{(recall Chapter 1 problems)}$$

$y = e^{rx}$ is a solution of (1) if and only if

$$r^2 + ar + b = 0 \qquad (2)$$

Equation (2) is called the **characteristic equation** of equation (1)

Note the correspondence:

Diff. Eqn: $y'' + ay' + by = 0$

Char. Eqn: $r^2 + ar + b = 0$

The solutions $y = e^{rx}$ of

$$y'' + ay' + by = 0$$

are determined by the roots of

$$r^2 + ar + b = 0.$$

RECALL: **Quadratic Equations**

General Case: $ar^2 + br + c = 0$

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Our Special Case: $r^2 + ar + b$

Roots: $r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

Example: Find a fundamental set of solutions of:

$$y'' - 6y' + 8y = 0$$

Characteristic equation:

Roots:

Solutions:

Wronskian $W(x) =$

Fundamental set:

There are **three cases:**

1. $r^2 + ar + b = 0$ has **two, distinct real roots**, $r_1 = \alpha$, $r_2 = \beta$.

2. $r^2 + ar + b = 0$ has **only one real root**, $r = \alpha$.

3. $r^2 + ar + b = 0$ has **complex conjugate roots**, $r_1 = \alpha + i\,\beta$, $r_2 = \alpha - i\,\beta$, $\beta \neq 0$.

Case I: Two, distinct real roots.

$r^2 + ar + b = 0$ has two distinct real roots:

$$r_1 = \alpha, \quad r_2 = \beta, \ \alpha \neq \beta.$$

Then

$$y_1(x) = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = e^{\beta x}$$

are solutions of $y'' + ay' + by = 0$.

$y_1 = e^{\alpha x}$ and $y_2 = e^{\beta x}$ are not constant multiples of each other

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Fundamental set: $\left\{e^{\alpha x},\ e^{\beta x}\right\}$

General solution:

$$y = C_1\, e^{\alpha x} + C_2\, e^{\beta x}$$

Example 1: Find the general solution of

$$y'' - 3y' - 10y = 0.$$

Example 2: Find a fundamental set of solutions of

$$y'' - 11y' + 28y = 0.$$

Case II: Exactly one real root.

Example: $y'' - 4y' + 4y = 0$

$r = \alpha$; (α is a **double root**). Then

$$y_1(x) = e^{\alpha x}$$

is one solution of $y'' + ay' + by = 0$.

We need a second solution which is not a constant multiple of y_1, i.e., is independent of y_1.

NOTE: In this case, the characteristic equation is

$$(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0$$

so the differential equation is

$$y'' - 2\alpha y' + \alpha^2 y = 0$$

$y = Ce^{\alpha x}$ is a solution for any constant C. Replace C by a function u *

which is to be determined, **if possible**, so that

$$y = u(x)e^{\alpha x}$$

is a solution of: $\quad y'' - 2\alpha\, y' + \alpha^2\, y = 0$

$$y = ue^{\alpha x}$$

$$y' = \alpha\, u\, e^{\alpha x} + e^{\alpha x}u'$$

$$y'' = \alpha^2 ue^{\alpha x} + 2\alpha e^{\alpha x}u' + e^{\alpha x}u''$$

*called "variation of parameters"

$y_1 = e^{\alpha x}$ and $y_2 = xe^{\alpha x}$ are **not** constant multiples of each other, $\{y_1, y_2\}$ is a fundamental set,

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$\begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + \alpha x e^{\alpha x} \end{vmatrix} = e^{2\alpha x} \neq 0$$

General solution:

$$y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}$$

Examples:

1. Find the general solution of

$$y'' + 6y' + 9y = 0.$$

2. Find the general solution of

$$y'' - 10y' + 25y = 0.$$

Case III: Complex conjugate roots.

$$r_1 = \alpha + i\,\beta, \quad r_2 = \alpha - i\,\beta, \quad \beta \neq 0$$

In this case $r_1 \neq r_2$ and so

$$u_1(x) = e^{(\alpha+i\beta)x} \quad u_2(x) = e^{(\alpha-i\beta)x}$$

are ind. solns. of $y'' + ay' + by = 0$

and

$$y = C_1\, e^{(\alpha+i\beta)x} + C_2\, e^{(\alpha-i\beta)x}$$

is the general solution. BUT, these are complex-valued functions!! **No good!!**

We want real-valued solutions!!

Recall from Calculus II:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^{2n}}{(2n)!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \pm \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Let $x = i\theta,$ and recall $i^2 = -1$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\,\theta - \frac{\theta^2}{2!} - i\,\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\,\frac{\theta^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\,\theta - i\,\frac{\theta^3}{3!} + i\,\frac{\theta^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\,\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

Relationships between the exponential function, sine and cosine

Euler's Formula: $\boxed{e^{i\theta} = \cos\,\theta + i\,\sin\,\theta}$

These formulas follow:

$$e^{-i\theta} = \cos\,\theta - i\,\sin\,\theta \qquad \big(\text{replace } \theta \text{ by } -\theta\big)$$

$$\cos\,\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \big(e^{i\theta} + e^{-i\theta}\big)$$

$$\sin\,\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \big(e^{i\theta} - e^{-i\theta}\big)$$

The Most Beautiful Equation

$e^{i\theta} = \cos\,\theta + i\,\sin\,\theta$

Set $\theta = \pi$:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

or

$$\boxed{e^{i\pi} + 1 = 0}$$

Now

$$u_1 = e^{(\alpha+i\,\beta)x} = e^{\alpha x} \cdot e^{i\beta x}$$

$$= e^{\alpha x}(\cos\beta x + i\ \sin\beta x)$$

$$= e^{\alpha x}\cos\beta x + i\, e^{\alpha x}\sin\beta x \qquad (1)$$

$$u_2 = e^{(\alpha-i\,\beta)x} = e^{\alpha x} \cdot e^{-i\beta x}$$

$$= e^{\alpha x}(\cos\beta x - i\ \sin\beta x)$$

$$= e^{\alpha x}\cos\beta x - i\, e^{\alpha x}\sin\beta x \qquad (2)$$

$$y_1 = \frac{u_1 + u_2}{2} = e^{\alpha x}\cos\beta x \qquad (1)+(2)$$

$$y_2 = \frac{u_1 - u_2}{2i} = e^{\alpha x}\sin\beta x \qquad (1)-(2)$$

y_1 and y_2 are **real-valued solutions**.

$$\{u_1 = e^{(\alpha+i\beta)x}, \quad u_2 = e^{(\alpha-i\beta)x}\}$$

transforms into

$$\{y_1 = e^{\alpha x}\cos\,\beta x, \quad y_2 = e^{\alpha x}\sin\,\beta x\}$$

y_1 and y_2 are not constant multiples of each other, $\{y_1, y_2\}$ is a fundamental set,

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \beta e^{2\alpha x} \neq 0$$

AND

$$y = C_1 \, e^{\alpha x} \cos \, \beta x + C_2 \, e^{\alpha x} \sin \, \beta x$$

is the general solution.

Examples: Find the general solution of

1. $y'' - 4y' + 13y = 0.$

2. $y'' + 6y' + 25y = 0.$

Important Special Case:

Example: $y'' + 4y = 0$

Characteristic equation: $r^2 + 4 = 0$

Roots: $2i$, $-2i$ that is: $0 + 2i$, $0 - 2i$

Solutions:

$$y_1 = e^{0x} \cos 2x = \cos 2x,$$

$$y_2 = e^{0x} \sin 2x = \sin 2x$$

In general: $y'' + \alpha^2 y = 0$

Characteristic equation: $r^2 + \alpha^2 = 0$

Roots: $\alpha i,\ -\alpha i$ that is: $0 + \alpha i,\ 0 - \alpha i$

Solutions:

$$y_1 = e^{0x} \cos \alpha x = \cos \alpha x$$

$$y_2 = e^{0x} \sin \alpha x = \sin \alpha x$$

That is: $y_1 = \cos \alpha x,\ y_2 = \sin \alpha x$

is a fundamental set.

Example: $y'' + 7y = 0$

Characteristic equation: $r^2 + 7 = 0$

Roots: $i\sqrt{7},\ -i\sqrt{7}$ that is:

$0 + i\sqrt{7},\ 0 - i\sqrt{7}$

Solutions:

$$y_1 = e^{0x} \cos \sqrt{7}\, x = \cos \sqrt{7}\, x,$$

$$y_2 = e^{0x} \sin \sqrt{7}\, x = \sin \sqrt{7}\, x$$

Comprehensive Examples:

I. Given the DE, find the solutions:

1. Find the general solution of

$$y'' - 2y' - 24y = 0.$$

2. Find a solution basis for

$$y'' - 10y' + 25y = 0.$$

3. Find a fundamental set of solutions of

$$y'' - 4y' + 8y = 0.$$

II. Given the solutions, find the DE

1. Find the differential equation that has

$$y = C_1 e^{2x} + C_2 e^{3x}$$

as its general solution. (See Chap 1, pg 39)

2. Find the differential equation that has

$$y = C_1 e^{-x} + C_2 e^{4x}$$

as its general solution. (c.f., Chap. 1 approach)

3. $y = 5xe^{-4x}$ is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?

4. $y = 2e^{2x}\sin 4x$ is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?

5. $y = C_1 e^{x} + C_2 e^{-2x}$.

6. $y = C_1 e^{2x} + C_2 x e^{2x}$

7. $y = C_1 \cos 3x + C_2 \sin 3x$.

8. $y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$.