Second Order Linear Differential Equations (Text: Section 3.1)

A second order linear differential equation is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where p, q, and f are continuous functions on some interval I.

The functions p and q are called the **coefficients** of the equation.

In applications, the function f is often called the **forcing function** (see Section 3.6).

"Linear"

Set L[y] = y'' + p(x)y' + q(x)y. Then, for any two twice differentiable functions $y_1(x)$ and $y_2(x)$, $L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$ and, for any constant c_{i} L[cy(x)] = cL[y(x)].That is, L is a linear differential operator.

L[y] = y'' + py' + qy

$$L[y_1 + y_2] =$$

$$(y_1 + y_2)'' + p (y_1 + y_2)' + q (y_1 + y_2)$$

= $y_1'' + y_2'' + p (y_1' + y_2') + q (y_1 + y_2)$
= $y_1'' + y_2'' + py_1' + py_2' + qy_1 + qy_2$
= $(y_1'' + py_1' + qy_1) + (y_2'' + py_2' + qy_2)$
= $L[y_1] + L[y_2]$

$$L[cy] = (cy)'' + p(cy)' + q(cy)$$

= $cy'' + pcy' + qcy = c(y'' + py' + qy)$
= $cL[y]$

THEOREM: Given the second order linear equation (1). Let a be any point on the interval I, and let α and β be any two real numbers. Then the initial-value problem

$$y'' + p(x) y' + q(x) y = f(x),$$

$$y(a) = \alpha, \ y'(a) = \beta$$

has a unique solution.

Homogeneous/Nonhomogeneous Equations

The linear differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$
 (1)

is **homogeneous**^{*} if the function f on the right side is 0 for all $x \in I$. That is,

$$y'' + p(x) y' + q(x) y = 0.$$

is a linear homogeneous equation.

*There is no relation between the term here and 1st order "homogeneous" equations in 2.4 If f is not the zero function on I, that is, if $f(x) \neq 0$ for some $x \in I$, then

$$y'' + p(x)y' + q(x)y = f(x)$$

is a linear nonhomogeneous equation.

Homogeneous Equations (Text, Section 3.2)

$$y'' + p(x) y' + q(x) y = 0$$
 (H)

where p and q are continuous functions on some interval I.

The zero function, y(x) = 0 for all $x \in I$, ($y \equiv 0$) is a solution of (H):

The zero solution $y \equiv 0$ is called the **trivial solution**. Any other solution is a **nontrivial** solution.

Recall, Example 8, Chap. 1, pg 20:

Find a value of r, if possible, such that $y = x^r$ is a solution of

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0.$$

Solutions:

 $y \equiv 0$ is a solution (trivial)

 $y_1 = x^{-1}, \quad y_2 = x^3$ are solutions

 $y = C_1 x^{-1} + C_2 x^3$ for any constants C_1, C_2

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THEOREM 1: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then

$$u(x) = y_1(x) + y_2(x)$$

is also a solution of (H).

The sum of any two solutions of **(H)** is also a solution of **(H)**. (Some call this property the *superposition principle*).

Proof:

 y_1 and y_2 are solutions. Therefore,

 $L[y_1] = 0$ and $L[y_2] = 0$

L is linear. Now

THEOREM 2: If y = y(x) is a solution of (H) and if *C* is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

Proof: y is a solution means L[y] = 0. L is linear:

Any constant multiple of a solution of (H) is also a solution of (H). **DEFINITION:** Let $y = y_1(x)$ and $y = y_2(x)$ be functions defined on some interval *I*, and let C_1 and C_2 be real numbers. The expression

$$C_1 y_1(x) + C_2 y_2(x)$$

is called a **linear combination** of y_1 and y_2 . Theorems 1 & 2 can be restated as:

THEOREM 3: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), and if C_1 and C_2 are any two real numbers, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).

NOTE: $y(x) = C_1 y_1(x) + C_2 y_2(x)$

is a two-parameter family which "looks like" the general solution.

Is it???

Some Examples from Chapter 1:

1.

$$y_1 = \cos 3x$$
 and $y_2 = \sin 3x$

are solutions of

$$y'' + 9y = 0$$
 (Chap 1, p. 46)

 $y = C_1 \cos 3x + C_2 \sin 3x$

is the general solution.

2. $y_1 = e^{-2x}$ and $y_2 = e^{4x}$

are solutions of

$$y'' - 2y' - 8y = 0$$
 (Chap 1, p. 54)

and

$$y = C_1 e^{-2x} + C_2 e^{4x}$$

is the general solution.

3. $y_1 = x$ and $y_2 = x^3$

are solutions of

$$y'' - \frac{3}{x}y' - \frac{3}{x^2}y = 0$$
 (Chap 1, p. 55)

and

$$y = C_1 x + C_2 x^3$$

is the general solution.

Example:
$$y'' - \frac{1}{x}y' - \frac{15}{x^2}y = 0$$

a. Solutions

$$y_1(x) = x^5, \quad y_2(x) = 3x^5$$

General solution:

$$y = C_1 x^5 + C_2(3x^5)$$
 ??

That is, is EVERY solution a linear combination of

$$y_1$$
 and y_2 ?

ANSWER: NO !!!

$$y = x^{-3}$$
 is a solution: (verify this)

AND

$$x^{-3} \neq C_1 x^5 + C_2 (3x^5)$$

since

$$C_1 x^5 + C_2 (3x^5) = (C_1 + 3C_2) x^5 = M x^5$$

CLEARLY, x^{-3} is **NOT** a constant multiple of x^5 . Now consider

$$y_1(x) = x^5, \ y_2(x) = x^{-3}$$

General solution: $y = C_1 x^5 + C_2 x^{-3}$?

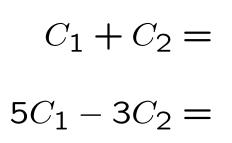
That is, is EVERY solution a linear combination of y_1 and y_2 ??

Let y = y(x) be **any** solution of the equation. Suppose

$$y(1) = y'(1) =$$

$$y = C_1 x^5 + C_2 x^{-3}$$
$$y' = 5C_1 x^4 - 3C_2 x^{-4}$$

At x = 1:



In general: Let

$$y = C_1 y_1(x) + C_2 y_2(x)$$

be a two-parameter family of solutions of (H). When is this the general solution of (H)?

EASY ANSWER: When y_1 and y_2 **ARE NOT CONSTANT MULTIPLES OF EACH OTHER.**

That is, y_1 and y_2 are independent of each other.

$$y = C_1 y_1(x) + C_2 y_2(x)$$

be a two parameter family of solutions of (H). Choose any number $a \in I$ and let u be any solution of (H).

Suppose
$$u(a) = \alpha$$
, $u'(a) = \beta$

Does the system of equations

$$C_1 y_1(a) + C_2 y_2(a) = \alpha$$

 $C_1 y'_1(a) + C_2 y'_2(a) = \beta$

have a unique solution??

DEFINITION: Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H). The function W defined by

 $W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$

is called the Wronskian of y_1, y_2 .

Determinant notation:

$$W(y_1, y_2) = y_1(x)y'_2(x) - y_2(x)y'_1(x)$$
$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}$$

Examples

1. Let $y_1 = x^{-3}$, $y_2 = x^5$. Then $W(y_1, y_2) = \begin{vmatrix} x^{-3} & x^5 \\ -3x^{-4} & 5x^4 \end{vmatrix}$ $=x^{-3}(5x^4) - x^5(-3x^{-4})$ =8x

$$= 8x$$

2. Let $y_1 = x^5$, $y_2 = 3x^5$. Then

$$W(y_1, y_2) = \begin{vmatrix} x^5 & 3x^5 \\ 5x^4 & 15x^4 \end{vmatrix}$$
$$= 15x^9 - 15x^9$$

= 0

Given two functions y_1 and y_2 .

Suppose $W(y_1, y_2) \equiv 0$

Set
$$u(x) = \frac{y_2(x)}{y_1(x)}$$
. Then

THEOREM 4: Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (H), and let W(x) be their Wronskian. Exactly one of the following holds:

(i) W(x) = 0 for all $x \in I$ ($W(x) \equiv 0$ on I); that is, y_1 and y_2 are constant multiples of each other, AND

$$y = C_1 y_1(x) + C_2 y_2(x)$$

IS NOT the general solution of (H) **OR**

(ii) $W(x) \neq 0$ for all $x \in I$, AND

$y = C_1 y_1(x) + C_2 y_2(x)$

IS the general solution of (H)

Fundamental Set; Solution Basis; Linearly Independent Set

DEFINITION: A pair of solutions

$$y = y_1(x), \quad y = y_2(x)$$

of equation (H) forms a **fundamental set of solutions** (also called a **solution basis** or a **linearly independent set**) if y_1 and y_2 are not constant multiples of each other. Equivalently,

 $W[y_1, y_2](x) \neq 0$ for all $x \in I$.

Homogeneous Equations with Constant Coefficients (Text, Section 3.3) Fact: In contrast to first order linear equations, there are **no general meth**-

ods for solving

$$y'' + p(x)y' + q(x)y = 0.$$
 (H)

But, there is a special case of (H) for which there is a solution method, namely when the equation has constant coefficients:

$$y'' + ay' + by = 0$$
 (1)

where a and b are constants.

Solutions: (1) has solutions of the form

$$y = e^{rx}$$
 (recall Chapter 1 problems)

 $y = e^{rx}$ is a solution of (1) if and only if

$$r^2 + ar + b = 0 \tag{2}$$

Note the correspondence:

Diff. Eqn:
$$y'' + ay' + by = 0$$

Char. Eqn: $r^2 + ar + b = 0$

The solutions $y = e^{rx}$ of

$$y'' + ay' + by = 0$$

are determined by the roots of

$$r^2 + ar + b = 0.$$

RECALL: Quadratic Equations

General Case: $ar^2 + br + c = 0$

Roots:
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Our Special Case: $r^2 + ar + b$

Roots:
$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Example: Find a fundamental set of solutions of:

$$y'' - 6y' + 8y = 0$$

Characteristic equation:

Roots:

Solutions:

Wronskian W(x) =

Fundamental set:

There are three cases:

1.
$$r^2 + ar + b = 0$$
 has two, distinct

real roots, $r_1 = \alpha$, $r_2 = \beta$.

2.
$$r^2 + ar + b = 0$$
 has only one real root, $r = \alpha$.

3. $r^2 + ar + b = 0$ has complex conjugate roots, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$. Case I: Two, distinct real roots.

 $r^2 + ar + b = 0$ has two distinct real roots:

$$r_1 = \alpha, \quad r_2 = \beta, \ \alpha \neq \beta.$$

Then

 $y_1(x) = e^{\alpha x}$ and $y_2(x) = e^{\beta x}$

are solutions of y'' + ay' + by = 0.

 $y_1 = e^{\alpha x}$ and $y_2 = e^{\beta x}$ are not constant multiples of each other

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Fundamental set:
$$\{e^{lpha x}, e^{eta x}\}$$

General solution:

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x}$$

Example 1: Find the general solution of

y'' - 3y' - 10y = 0.

Example 2: Find a fundamental set of solutions of

$$y'' - 11y' + 28y = 0.$$

Case II: Exactly one real root.

Example: y'' - 4y' + 4y = 0

 $r=lpha;~(lpha~~{
m is~a~}{
m double~root}).$ Then $y_1(x)=e^{lpha x}$

is one solution of y'' + ay' + by = 0.

We need a second solution which is not a constant multiple of y_1 , i.e., is independent of y_1 . **NOTE:** In this case, the characteristic equation is

$$(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0$$

so the differential equation is

$$y'' - 2\alpha y' + \alpha^2 y = 0$$

 $y = Ce^{\alpha x}$ is a solution for any constant C. Replace C by a function u * which is to be determined, if possible, so that

$$y = u(x)e^{\alpha x}$$

is a solution of: $y'' - 2\alpha y' + \alpha^2 y = 0$

$$y = ue^{\alpha x}$$
$$y' = \alpha u e^{\alpha x} + e^{\alpha x} u'$$
$$y'' = \alpha^2 ue^{\alpha x} + 2\alpha e^{\alpha x} u' + e^{\alpha x} u''$$

*called "variation of parameters"

 $y_1 = e^{\alpha x}$ and $y_2 = xe^{\alpha x}$ are **not** constant multiples of each other, $\{y_1, y_2\}$ is a fundamental set,

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$\begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + \alpha xe^{\alpha x} \end{vmatrix} = e^{2\alpha x} \neq 0$$

General solution:

$$y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}$$

Examples:

1. Find the general solution of

$$y'' + 6y' + 9y = 0.$$

2. Find the general solution of

y'' - 10y' + 25y = 0.

Case III: Complex conjugate roots.

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \beta \neq 0$$

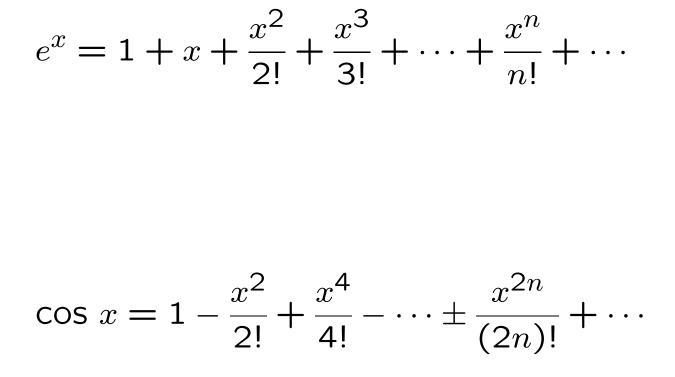
In this case $r_1 \neq r_2$ and so

 $u_1(x) = e^{(\alpha + i\beta)x}$ $u_2(x) = e^{(\alpha - i\beta)x}$

are ind. solns. of y'' + ay' + by = 0and

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

is the general solution. BUT, these are complex-valued functions!! **No good!! We want real-valued solutions!!** 48 Recall from Calculus II:



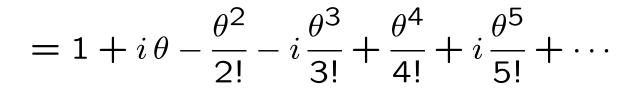
sin
$$x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \pm \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

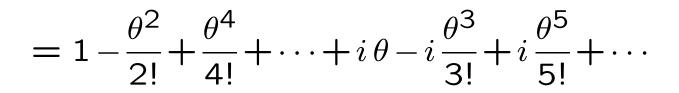
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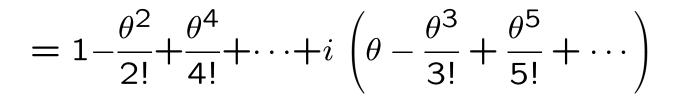
$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots + \frac{x^{n}}{n!} + \dots$$

Let $x = i\theta$, and recall $i^2 = -1$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$







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Relationships between the exponen-

tial function, sine and cosine

Euler's Formula:
$$e^{i\theta} = \cos \theta + i \sin \theta$$

These formulas follow:

$$e^{-i heta}= \cos\, heta-i\,\sin\, heta$$
 (replace $heta$ by $_{- heta})$

$$\cos \theta = rac{e^{i\theta} + e^{-i\theta}}{2}$$
 $(e^{i\theta} + e^{-i\theta})$

$$\sin \theta = rac{e^{i heta} - e^{-i heta}}{2i}$$
 $(e^{i heta} - e^{-i heta})$

The Most Beautiful Equation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Set $\theta = \pi$:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

or

$$e^{i\pi} + 1 = 0$$

Now

$$u_{1} = e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x}$$
$$= e^{\alpha x} (\cos \beta x + i \sin \beta x)$$
$$= e^{\alpha x} \cos \beta x + i e^{\alpha x} \sin \beta x \qquad (1)$$

$$u_{2} = e^{(\alpha - i\beta)x} = e^{\alpha x} \cdot e^{-i\beta x}$$
$$= e^{\alpha x} (\cos \beta x - i \sin \beta x)$$
$$= e^{\alpha x} \cos \beta x - i e^{\alpha x} \sin \beta x \qquad (2)$$

$$y_1 = \frac{u_1 + u_2}{2} = e^{\alpha x} \cos \beta x$$
 (1)+(2)

$$y_2 = \frac{u_1 - u_2}{2i} = e^{\alpha x} \sin \beta x$$
 (1) - (2)

 y_1 and y_2 are **real-valued solutions**.

$$\{u_1 = e^{(\alpha + i\beta)x}, \quad u_2 = e^{(\alpha - i\beta)x}\}$$

transforms into

$$\{y_1 = e^{\alpha x} \cos \beta x, \quad y_2 = e^{\alpha x} \sin \beta x\}$$

 y_1 and y_2 are not constant multiples of each other, $\{y_1, y_2\}$ is a fundamental set,

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \beta e^{2\alpha x} \neq 0$$

AND

 $y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

is the general solution.

Examples: Find the general solution of

1. y'' - 4y' + 13y = 0.

2. y'' + 6y' + 25y = 0.

Important Special Case:

Example: y'' + 4y = 0

Characteristic equation: $r^2 + 4 = 0$

Roots: 2i, -2i that is: 0 + 2i, 0 - 2i

Solutions:

$$y_1 = e^{0x} \cos 2x = \cos 2x,$$

$$y_2 = e^{0x} \sin 2x = \sin 2x$$

In general: $y'' + \alpha^2 y = 0$

Characteristic equation: $r^2 + \alpha^2 = 0$

Roots: αi , $-\alpha i$ that is: $0 + \alpha i$, $0 - \alpha i$

Solutions:

$$y_1 = e^{0x} \cos \alpha x = \cos \alpha x$$

$$y_2 = e^{0x} \sin \alpha x = \sin \alpha x$$

That is: $y_1 = \cos \alpha x$, $y_2 = \sin \alpha x$

is a fundamental set.

Example: y'' + 7y = 0

Characteristic equation: $r^2 + 7 = 0$

Roots:
$$i\sqrt{7}$$
, $-i\sqrt{7}$ that is:
 $0 + i\sqrt{7}$, $0 - i\sqrt{7}$

Solutions:

$$y_1 = e^{0x} \cos\sqrt{7} \, x = \cos\sqrt{7} \, x,$$

$$y_2 = e^{0x} \sin \sqrt{7} \, x = \sin \sqrt{7} \, x$$

Comprehensive Examples:

- I. Given the DE, find the solutions:
- 1. Find the general solution of

$$y'' - 2y' - 24y = 0.$$

2. Find a solution basis for

$$y'' - 10y' + 25y = 0.$$

3. Find a fundamental set of solutions of

$$y'' - 4y' + 8y = 0.$$

II. Given the solutions, find the DE

Find the differential equation that
 has

$$y = C_1 e^{2x} + C_2 e^{3x}$$

as its general solution. (See Chap 1, pg 39)

2. Find the differential equation that has

$$y = C_1 e^{-x} + C_2 e^{4x}$$

as its general solution. (c.f., Chap. 1 approach)

3. $y = 5xe^{-4x}$ is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

 $\mathbf b.$ What is the general solution?

4. $y = 2e^{2x} \sin 4x$ is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

 \mathbf{b} . What is the general solution?

5.
$$y = C_1 e^x + C_2 e^{-2x}$$
.

6. $y = C_1 e^{2x} + C_2 x e^{2x}$

7. $y = C_1 \cos 3x + C_2 \sin 3x$.

8. $y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$.