Section 3.4. Second Order Nonhomogeneous Equations (Text, Section 3.4)

$$\boxed{y'' + p(x)y' + q(x)y = f(x)} \qquad \text{(N)}$$

The **corresponding homogeneous** equation

$$\boxed{y'' + p(x)y' + q(x)y = 0} \qquad \text{(H)}$$

is called the **reduced equation** of (N).

You will see that these two equations are closely connected.

Basic Results

THEOREM 1: If $z = z_1(x)$ and $z = z_2(x)$ are solutions of equation (N), then

$$y(x) = z_1(x) - z_2(x)$$

is a solution of equation (H).

Proof:

THEOREM 2: Let $\{y_1(x),\ y_2(x)\}$ be a fundamental set of solutions of the reduced equation (H), and let $z = z(x)$ be a particular solution of (N). Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + z(x)$$

is the general solution of (N).

Proof:

Conclusion: The general solution of (N) consists of **the general solution of the reduced equation (H)** ***plus*** **a particular solution z of (N):**

$$y = \underbrace{C_1 y_1(x) + C_2 y_2(x)} + \underbrace{z(x)}.$$

gen soln (H) $\quad + \quad$ part soln (N)

Example 1. $z_1(x) = 3x^2 + x\ln x$, $z_2(x) = x\ln x - 2x^2$ are solutions of

$$y'' + p(x)y' + q(x)y = f(x)$$

$y_1(x) = x^4$ is a solution of the reduced equation. What is the general solution the equation?

Example 2. $z_1(x) = 2x^2 + 2\cos 2x$, $z_2(x) = x^2 + 2\cos 2x$, $z_3(x) = x^3 + 2x^2 + 2\cos 2x$

are solutions of

$$y'' + p(x)y' + q(x)y = f(x)$$

The general solution of the equation is:

 # To find the general solution of (N):

$$y'' + p(x)y' + q(x)y = f(x)$$

you need to find:

1. a fundamental set of solutions y_1, y_2 of the reduced equation (H), and

2. a particular solution z of (N).

THEOREM 3: (Superposition Principle)

Given the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) + g(x).$$

If $z = z_f(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f(x)$$

and $z = z_g(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = g(x),$$

then

$$z(x) = z_f(x) + z_g(x)$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) + g(x).$$

Proof:

I. Variation of Parameters

Recall from 3.2: Let $y = y_1(x)$ and $y = y_2(x)$ be independent solutions of the reduced equation (H) and let

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0$$

be their Wronskian.

Then

$$y = C_1 y_1(x) + C_2 y_2(x)$$

is the general solution of (H).

Set

$$z(x) = u(x)y_1(x) + v(x)y_2(x)$$

where u and v are to be determined

so that z is a solution of (N).

$$z = uy_1 + vy_2$$

$$z' = uy_1' + y_1u' + vy_2' + y_2v'$$

Set $y_1u' + y_2v' = 0$. Then we have

$$z = uy_1 + vy_2$$

$$z' = uy_1' + vy_2' \quad \text{and}$$

$$z'' = uy_1'' + y_1'u' + vy_2'' + y_2'v'$$

Substitute z, z', z'' into the differential equation $y'' + py' + qy = f$.

$$\left(uy_1'' + y_1'u' + vy_2'' + y_2'v'\right) + p\left(uy_1' + vy_2'\right)$$

$$+ \; q\left(uy_1 + vy_2\right) = f$$

Rearrange the terms:

$$u\left(y_1'' + py_1' + qy_1\right) + v\left(y_2'' + py_2' + qy_2\right)$$

$$+ \; y_1'u' + y_2'v' = f$$

which reduces to $y_1'u' + y_2'v' = f$.

We now have two equations in the two unknowns u' and v':

$$y_1 u' + y_2 v' = 0$$

$$y_1' u' + y_2' v' = f$$

Solve for u':

$$\left(y_1 y_2' - y_2 y_1'\right) u' = -y_2 f$$

or

$$W u' = -y_2 f$$

so

$$u' = \frac{-y_2 f}{W} \quad \text{and} \quad \boxed{u = \int \frac{-y_2 f}{W}\, dx;}$$

Similarly, solve for v':

$$y_1 u' + y_2 v' = 0$$

$$y_1' u' + y_2' v' = f$$

Solve for v':

$$\left(y_1 y_2' - y_2 y_1'\right) v' = y_1 f$$

or

$$W v' = y_1 f$$

so

$$v' = \frac{y_1 f}{W} \quad \text{and} \quad \boxed{v = \int \frac{y_1 f}{W}\, dx;}$$

Therefore,

$z(x) =$

$$\boxed{y_1(x) \int \frac{-y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx}$$

is a particular solution of the nonhomogeneous equation (N).

Note: We used two independent solutions y_1, y_2 of the reduced equation to "construct" a solution of the nonhomogeneous equation.

Conclusion: We can solve **any** second order linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

provided we can find two linearly independent solutions y_1, y_2 of its reduced equation

$$y'' + p(x)y' + q(x)y = 0.$$

Examples:

1. $\left\{ y_1(x) = x^2,\ y_2(x) = x^4 \right\}$ is a fundamental set of solutions of

$$y'' - \frac{5}{x}\,y' + \frac{8}{x^2}\,y = 0 \qquad (x \neq 0)$$

Find a particular solution z of the nonhomogeneous equation.

$$y'' - \frac{5}{x}\,y' + \frac{8}{x^2}\,y = 4x^3$$

$$y_1 = x^2,\ y_2 = x^4$$

$$W[y_1, y_2] =$$

2. Find the general solution of

$$y'' - 4y' + 4y = \frac{e^{2x}}{x} \qquad (x \neq 0)$$

$y_1 = e^{2x},\ y_2 = xe^{2x}$

$W[y_1, y_2] =$

3. Find the general solution of

$$y'' + y' - 6y = 3e^{2x}$$

$y_1 = e^{-3x},\ y_2 = e^{2x}$

$W[y_1, y_2] =$

4. Find the general solution of

$$y'' + 4y = 2\tan 2x$$

$y_1 = \cos 2x,\ y_2 = \sin 2x$

$W[y_1, y_2] =$

II. Undetermined Coefficients aka "Guessing" (Text, Section 3.5)

NOTE: THIS METHOD CAN BE USED ONLY WHEN:

1. The de has constant coefficients

2. f is an "exponential" function

That is: $y'' + ay' + by = f(x)$ where $a,\ b$ are constants, and f is an "exponential" function.

Basic Exponential Functions:

$$e^{\gamma x}$$

$$\cos \delta x, \qquad \sin \delta x$$

$$e^{\gamma x} \cos \delta x, \qquad e^{\gamma x} \sin \delta x$$

There are three basic cases to consider:

1. $y'' + ay' + by = \alpha\, e^{rx}$

2. $y'' + ay' + by = \alpha\, \cos \delta x + \beta\, \sin \delta x$

3. $y'' + ay' + by = \alpha\, e^{\gamma x} \cos \delta x + \beta\, e^{\gamma x} \sin \delta x$

Recall Problem 4, EMCF 2:

Find A so that $z = Ae^{-2x}$ is a solution of

$$y'' - 5y' + 6y = 5e^{-2x}.$$

Case 1: If $y'' + ay' + by = \alpha e^{rx}$

Set $z(x) = Ae^{rx}$ and find A.

Note: The coefficient A is called an **undetermined coefficient**.

Example 1: Find a particular solution z of

$$y'' - 5y' + 6y = 7e^{-4x}.$$

Also, give the general solution of the equation.

Set $z = Ae^{-4x}$ where A is to be determined;

$$z = Ae^{-4x}$$

$$z' = -4Ae^{-4x}$$

$$z'' = 16Ae^{-4x}$$

Answer: $z = \frac{1}{6}\, e^{-4x}$.

The general solution of the differential equation is:

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{6}\, e^{-4x}.$$

Note: If $L[y] = y'' + ay' + by$, then

$$L[Ae^{rx}] = A\left(r^2 + ar + b\right) e^{rx} = Ke^{rx}$$

That is, $L[Ae^{rx}]$ is a constant multiple of e^{rx}. In Example 1,

$$L\left[Ae^{-4x}\right] = 42Ae^{-4x}.$$

Example 2: Find a particular solution

z of

$$y'' + 2y' - 3y = 9e^{-2x}.$$

and give the general solution.

$$z = Ae^{-2x}$$

$$z' = -2Ae^{-2x}$$

$$z'' = 4Ae^{-2x}$$

Case 2: $y'' + ay' + by = \alpha \cos \delta x,$

or $y'' + ay' + by = \beta \sin \delta x,$

or $y''+ay'+by = \alpha \cos \delta x+\beta \sin \delta x,$

Example: $y'' - 2y' + y = 5 \cos 2x$

Set $z = A \cos 2x$???

$$z = A \cos 2x \quad ???$$

$$z' = -2A \sin 2x$$

$$z'' = -4A \cos 2x$$

Note: If $L[y] = y'' + ay' + by$, then

$$L[A\cos\beta x] = K\cos\beta x + M\sin\beta x$$

That is, $L[A\cos\beta x]$ involves BOTH cosine and sine. Similarly for $L[B\sin\beta x]$ and $L[A\cos\beta x + B\sin\beta x]$

Therefore, if $f(x) = c\cos\beta x$ or

$f(x) = d\sin\beta x$ or

$f(x) = c\,\cos\beta x + d\,\sin\beta x$

set $\quad z(x) = A \cos \beta x + B \sin \beta x$

where A, B are to be determined

Note: A and B are *undetermined co-efficients*.

Example 3: Find a particular solution z of

$$y'' - 2y' + y = 5\cos 2x.$$

and give the general solution of the equation.

Set $\qquad z = A\cos 2x + B\sin 2x$

$$z = A\cos 2x + B\sin 2x$$

$$z' = -2A\sin 2x + 2B\cos 2x$$

$$z'' = -4A\cos 2x - 4B\sin 2x$$

Answer: $z = \dfrac{3}{5}\cos 2x - \dfrac{4}{5}\sin 2x.$

The general solution of the differential equation is:

$$y = C_1 e^x + C_2 x e^x + \frac{3}{5}\cos 2x - \frac{4}{5}\sin 2x.$$

Example 4: Find a particular solution z of

$$y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x - 3e^{2x}$$

Set

$$z = A\cos 3x + B\sin 3x + Ce^{2x}$$

where A, B, C are to be determined.

$$y'' - 2y' + 5y = 2\cos\ 3x - 4\sin\ 3x - 3e^{2x}$$

Set $\quad z = A\cos 3x + B\sin 3x + Ce^{2x}$

$$z = A\cos 3x + B\sin 3x + Ce^{2x}$$

$$z' = -3A\sin 3x + 3B\cos 3x + 2Ce^{2x}$$

$$z'' = -9A\cos 3x - 9B\sin 3x + 4Ce^{2x}$$

Case 3: If $f(x) = ce^{\alpha x}\cos\beta x,\;\; de^{\alpha x}\sin\beta x$

or $\quad ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x$

set $\quad z(x) = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$

where A, B are to be determined.

Example 5: Find a particular solution

z of

$$y'' + 9y = 4e^x \sin 2x.$$

Set $z = Ae^x \cos 2x + Be^x \sin 2x$

Answer: $z = -\frac{4}{13}\, e^x \cos 2x + \frac{6}{13}\, e^x \sin 2x.$

Example 6: Find a particular solution

z of

$$y'' - 2y' + y = 3e^{3x} - 5\sin 2x.$$

$$z = Ae^{3x} + B\cos 2x + C\sin 2x$$

$$z' = 3Ae^{3x} - 2B\sin 2x + 2C\cos 2x$$

$$z'' = 9Ae^{3x} - 4B\cos 2x - 4C\sin 2x$$

Example 7: Find a particular solution z of

$$y'' + y' - 6y = 3e^{2x}.$$

$$z = Ae^{2x}$$

$$z' = 2Ae^{2x}$$

$$z'' = 4Ae^{2x}$$

A BIG Difficulty: The trial solution z is a solution of the reduced equation.

In this case, $y_1 = e^{-3x}$ and $y_2 = e^{2x}$ are solutions of the reduced equation

$$y'' + y' - 6y = 0$$

From Example 3, Section 3.4:

$$z = \frac{3}{5}x\,e^{2x}$$

Example 7 continued: Find a particular solution z of

$$y'' + y' - 6y = 3e^{2x}.$$

Reduced equation: $\quad y'' + y' - 6y = 0.$

Solutions: $y_1 = e^{2x}, \; y_2 = e^{-3x}$

Set $\quad z = Ae^{2x}$? $\quad$ NO!! This satisfies the reduced equation, so $L[Ae^{2x}] = 0$.

Set $\quad z = Axe^{2x}$.

$$y'' + y' - 6y = 3e^{2x}.$$

$$z = Axe^{2x}$$

$$z' = Ae^{2x} + 2Axe^{2x}$$

$$z'' = 4Ae^{2x} + 4Axe^{2x}$$

Example 8: Find a particular solution of

$$y'' - 2y' - 15y = 6e^{-3x}$$

Reduced equation: $y'' - 2y' - 15y = 0$

Solutions: $y_1 = e^{5x},\ y_2 = e^{-3x}$

$$z = Axe^{-3x}$$

$$z' = Ae^{-3x} - 3Axe^{-3x}$$

$$z'' = -6Ae^{-3x} + 9Axe^{-3x}$$

Example 9: Find a particular solution

z of

$$y'' + 4y = 2\cos 2x.$$

Reduced equation: $y'' + 4y = 0$

Solutions: $y_1 = \cos 2x,\ y_2 = \sin 2x$

Set $\quad z = A\cos 2x + B\sin 2x \quad$??

NO!! $\qquad L[z] = 0$

$$y'' + 4y = 2\cos 2x$$

$$z = Ax\cos 2x + Bx\sin 2x$$

$$z' = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x$$

$$z'' = -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x$$

Example 10: Find a particular solution z of

$$y'' + 6y' + 9y = 4e^{-3x}.$$

Reduced equation: $y'' + 6y' + 9y = 0$

Solutions: $y_1 = e^{-3x}$, $y_2 = xe^{-3x}$

Set $z = Ae^{-3x}$?

Set $z = Axe^{-3x}$?

$$y'' + 6y' + 9y = 4e^{-3x}$$

$$z = Ax^2e^{-3x}$$

$$z' = 2Axe^{-3x} - 3Ax^2e^{-3x}$$

$$z'' = 2Ae^{-3x} - 12Axe^{-3x} + 9Ax^2e^{-3x}$$

Example 11: Find a particular solution z of

$$y'' - 2y' - 8y = -3e^{-2x} + 6$$

Reduced equation: $y'' - 2y' - 8y = 0$

Solutions: $y_1 = e^{-2x}$, $y_2 = e^{4x}$

Set $z = Ae^{-2x} + B$??

$$y'' - 2y' - 8y = -3e^{-2x} + 6$$

$$z = Axe^{-2x} + B$$

$$z' = Ae^{-2x} - 2Axe^{-2x}$$

$$z'' = -4Ae^{-2x} + 4Axe^{-2x}$$

Example 12: Find a particular solution z of

$$y'' - 3y' = 4e^{3x} + 2$$

Reduced equation: $y'' - 3y' = 0$

Solutions: $y_1 = e^{3x},\ y_2 = e^{0x} = 1$

Set $z = Ae^{3x} + B = Ae^{3x} + Be^{0x}$??

$$y'' - 3y' = 4e^{3x} + 2$$

$$z = Axe^{3x} + Bx$$

$$z' = Ae^{3x} + 3Axe^{3x} + B$$

$$z'' = 6Ae^{3x} + 9Axe^{3x}$$

Answers:

$$z_8 = -\frac{3}{4}\, x\, e^{-3x}$$

$$z_9 = \frac{1}{2}\, x \sin\, 2x$$

$$z_{10} = 2x^2\, e^{-3x}$$

$$z_{11} = \frac{1}{2} x e^{-2x} - \frac{3}{4}$$

$$z_{12} = \frac{4}{3}\, x\, e^{3x} - \frac{2}{3}\, x$$

The Method of Undetermined Coefficients

A. Applies **only** to equations of the form

$$y'' + ay' + by = f(x)$$

where a, b are constants and f is an "exponential" function.

c.f. Variation of Parameters which can be applied to *any* linear nonhomogeneous equation.

I. Basic Case: If:

- $f(x) = ae^{rx}$ **set** $\boldsymbol{z = Ae^{rx}}$.

- $f(x) = c\cos\beta x,\ d\sin\beta x,$ or

$$c\cos\beta x + d\sin\beta x,$$

set $z = A\cos\beta x + B\sin\beta x$.

- $f(x) = ce^{\alpha x}\cos\beta x, \quad de^{\alpha x}\sin\beta x$ or

$$ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x,$$

set $z = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$.

BUT:

- If z satisfies the reduced equation, use xz;

- if xz also satisfies the reduced equation, then x^2z will give a particular solution.

II. General Case:

- If

$$f(x) = p(x)e^{rx}$$

where p is a polynomial of degree n,

then

set $z = P(x)e^{rx}$

where P is a polynomial of degree n

with undetermined coefficients.

Example 1: Find a particular solution

of $\quad y'' - 2y' - 8y = (4x + 5)e^{2x}.$

Set $\; z = (Ax + B)e^{2x}.$

Example 2: Find a particular solution

of $\quad y'' - 3y' + 2y = (2x^2 - 1)e^{-x}$.

Set $z = (Ax^2 + Bx + C)e^{-x}$.

$z = \left(\frac{1}{3}x^2 + \frac{5}{9}x + \frac{5}{27}\right) e^{-x}$.

- If

$$f(x) = p(x)\cos \beta x + q(x)\sin \beta x$$

where p, q are polynomials, then

set $z = P(x)\cos \beta x + Q(x)\sin \beta x$

where P, Q are polynomials of degree n with undetermined coefficients, $n =$ max degree of p and q.

Example 3:

$$y'' - 2y' - 3y = 3 \cos x + (x - 2) \sin x.$$

Set $z = (Ax+B) \cos x + (Cx+D) \sin x.$

Example 3 continued

$$z = \left(\frac{1}{10}x - \frac{47}{50}\right)\cos x - \left(\frac{1}{5}x - \frac{2}{25}\right)\sin x.$$

- If

$$f(x) = p(x)e^{\alpha x}\cos\,\beta x + q(x)e^{\alpha x}\sin\,\beta x$$

where $p,\ q$ are polynomials of degree n, then

set $z = P(x)e^{\alpha x}\cos\,\beta x+Q(x)e^{\alpha x}\sin\,\beta x$

where P, Q are polynomials of degree n with undetermined coefficients.

Example 4: $y'' + 4y = 2x\,e^x \cos x.$

Set

$$z = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$$

$$z = \frac{1}{25}(10x - 7)e^x \cos x + \frac{1}{25}(5x - 1)e^x \sin x.$$

Example 5: Find a particular solution

of $\quad y'' - 2y' - 8y = (4x + 5)e^{-2x}.$

Set $\; z = (Ax + B)e^{-2x}$????

BUT: Warning!!!

- If any part of z satisfies the reduced equation, try xz;

- if any part of xz also satisfies the reduced equation, then $x^2 z$ will give a particular solution.

Examples:

1. Give the form of a particular solution of

$$y'' - 4y' - 5y = 2\cos 3x - 5e^{5x} + 4.$$

2. Give the form of a particular solution of

$$y'' + 8y' + 16y = 2x - 1 + 7e^{-4x}.$$

3. Give the form of a particular solution of

$$y'' + y = 4\,\text{sin}\ x - \text{cos}\ 2x + 2e^{2x}.$$

4. Give the form of the general solution of

$$y'' + 9y = -4\cos 2x + 3\sin 2x$$

5. Give the form of a particular solution of

$$y'' + 9y = -4\cos 3x + 3\sin 2x$$

6. Give the form of the general solution of

$$y'' + 4y' + 4y = 4xe^{-2x} + 3$$

7. Give the form of the general solution of

$$y'' + 4y' + 4y = 4e^{-2x} \sin 2x + 3x$$

8. Give the form of the general solution of

$$y'' + 4y' = 5e^{-4x} + 4\sin 2x + 3$$

9. Give the form of a particular solution of

$$y'' + 2y' + 10y = 2e^{3x}\sin x + 4e^{3x}$$

10. Give the form of the general solution of

$$y'' + 2y' + 10y = 2e^{-x} \sin 3x + 2e^{-x}$$

11. Give the form of a particular solution of

$$y'' - 2y' - 8y = 2\cos 3x - (3x+1)e^{-2x} - 4$$

12. Give the form of a particular solution of

$$y'' - 2y' - 8y = 2\cos 3x - 3xe^{-2x} - 3x$$

13. Find the general solution of

$$y'' - 4y' + 4y = 4\sin 2x + \frac{e^{2x}}{x}$$

Summary: Solve

$$y'' + p(x)y' + q(x)y = f(x)$$

1. Variation of parameters:

- Can be applied to **any** linear nonhomogeneous equations, but
- **requires** a fundamental set of solutions of the reduced equation.

2. Undetermined coefficients:

- Is limited to linear nonhomogeneous equations with constant coefficients, and

- f must be an "exponential function,"

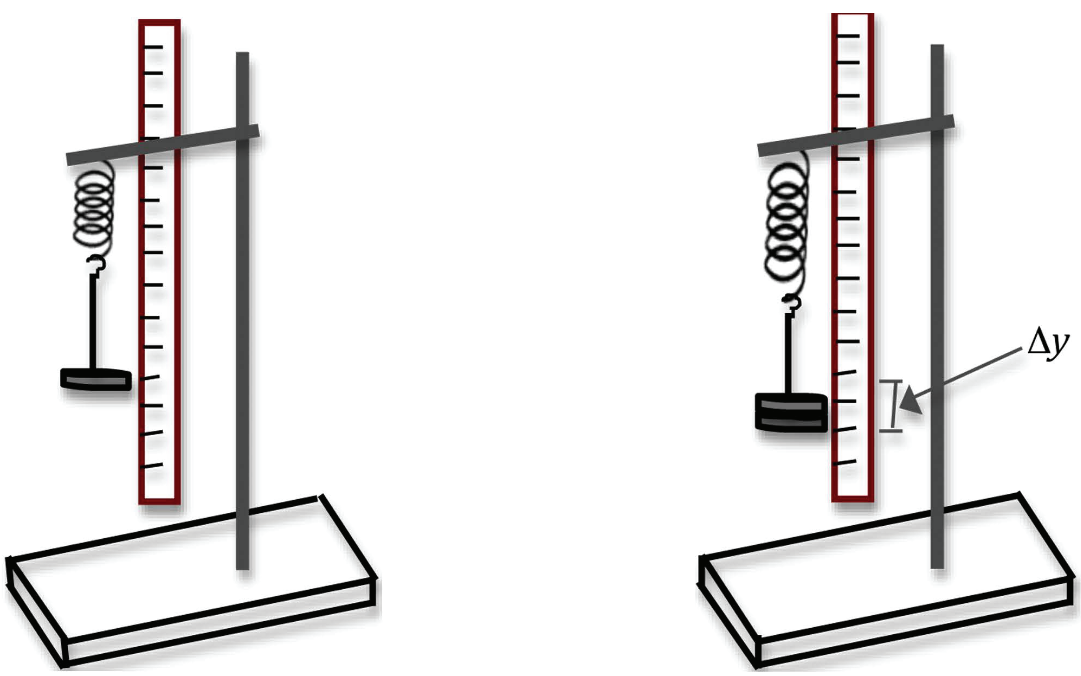
$f(x) = ae^{rx}$, $f(x) = c\cos\beta x + d\sin\beta x$,

$$f(x) = ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x,$$

or $p(x)f(x)$ p a polynomial.

In cases where both methods are applicable, the method of undetermined coefficients is usually more efficient and, hence, the preferable method.

Section 3.6. Vibrating Mechanical Systems (Text, Section 3.6)



I. Free Vibrations (Simple Harmonic Motion)

Hooke's Law: The restoring force of a spring is proportional to the displacement:

$$F = -ky, \;\; k > 0.$$

Newton's Second Law: Force equals mass times acceleration:

$$F = ma = m\frac{d^2y}{dt^2}.$$

Mathematical model:

$$m\frac{d^2y}{dt^2} = -ky$$

which can be written

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad (\text{Recall Section 3.3})$$

where $\omega = \sqrt{k/m}$.

The constant ω (omega) is called the **natural frequency** of the system.

Recall: Period $T = \dfrac{2\pi}{\omega}$.

The general solution of this equation is:

$$y = C_1 \sin \omega t + C_2 \cos \omega t$$

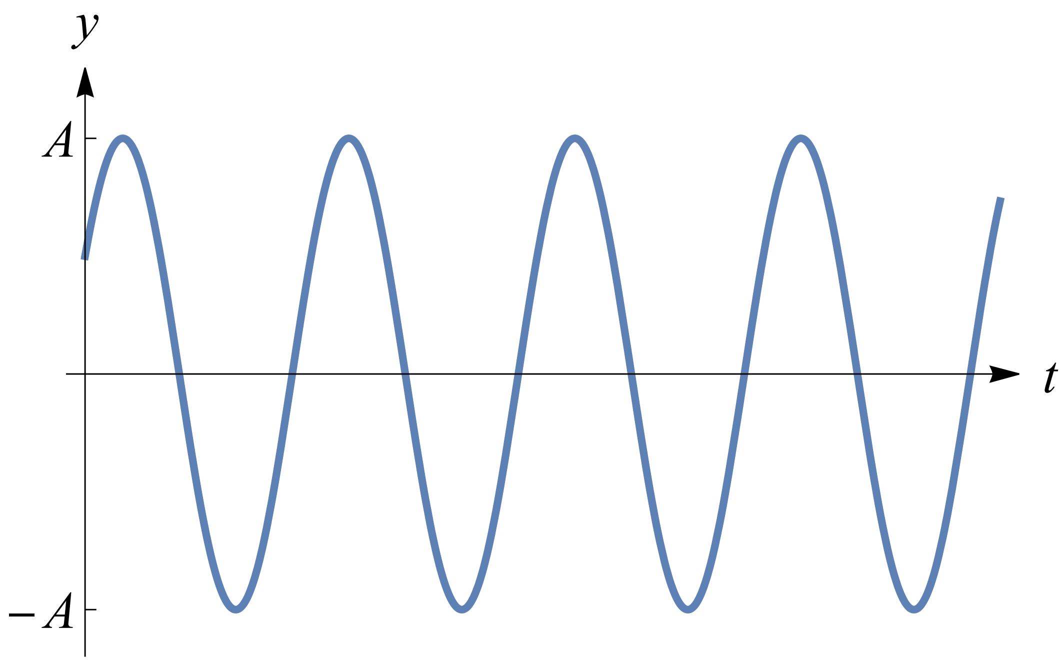
which can be written

$$y = A \sin(\omega t + \phi).$$

A is called the **amplitude**, ϕ is called the **phase shift**.

$$y = C_1 \sin \omega t + C_2 \cos \omega t$$

$$y = A\sin(\omega t + \phi)$$



Example. An object is in simple harmonic motion. Find an equation for the motion given that the period is $\frac{1}{4}\pi$ and, at time $t = 0$, $y = 1$, $y' = 0$. What is the natural frequency? What is the amplitude?

II. Forced Free Vibrations

Apply an external force G to the freely vibrating system

Force Equation:

$$F = -ky + G.$$

Mathematical Model:

$$my'' = -ky + G \qquad \text{or} \qquad y'' + \frac{k}{m}y = \frac{G}{m},$$

a nonhomogeneous equation.

A periodic external force:

$$G = a \text{ cos } \gamma t, \quad a, \ \gamma > 0 \text{ const.}$$

Force Equation:

$$F = -ky + a \text{ cos } \gamma t$$

Mathematical Model:

$$y'' + \frac{k}{m} y = \frac{a}{m} \text{ cos } \gamma t$$

$$y'' + \omega^2 y = \alpha \text{ cos } \gamma t$$

where $\omega = \sqrt{k/m}, \quad \alpha = \dfrac{a}{m}.$

ω is called the **natural frequency** of the system,

γ is called the **applied frequency**.

Case 1: $\gamma \neq \omega$.

$$y'' + \omega^2 y = \alpha \cos \gamma t$$

General solution, reduced equation:

$$y = C_1 \cos \omega t + C_2 \sin \omega t = A \sin(\omega t + \phi_0).$$

Form of particular solution (undetermined coefficients):

$$z = A \cos \gamma t + B \sin \gamma t.$$

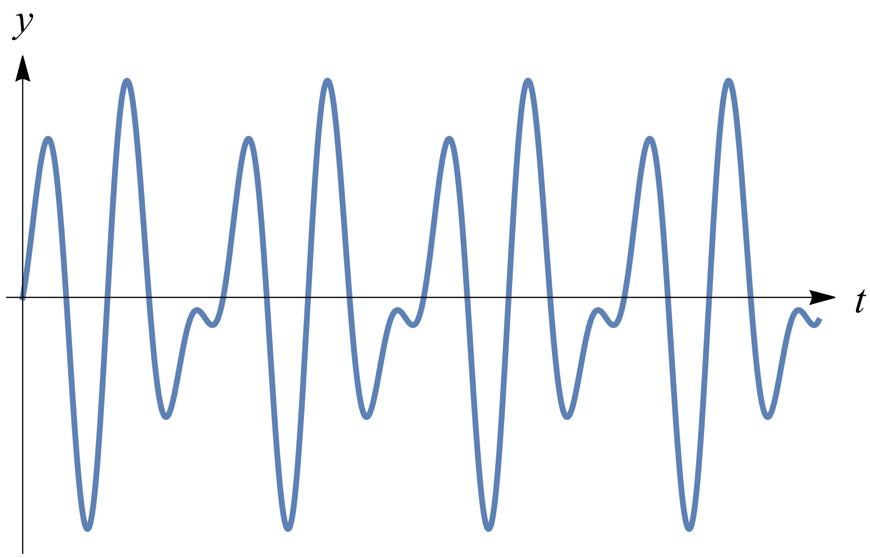
A particular solution:

$$z = \frac{\alpha}{\omega^2 - \gamma^2} \cos \gamma t.$$

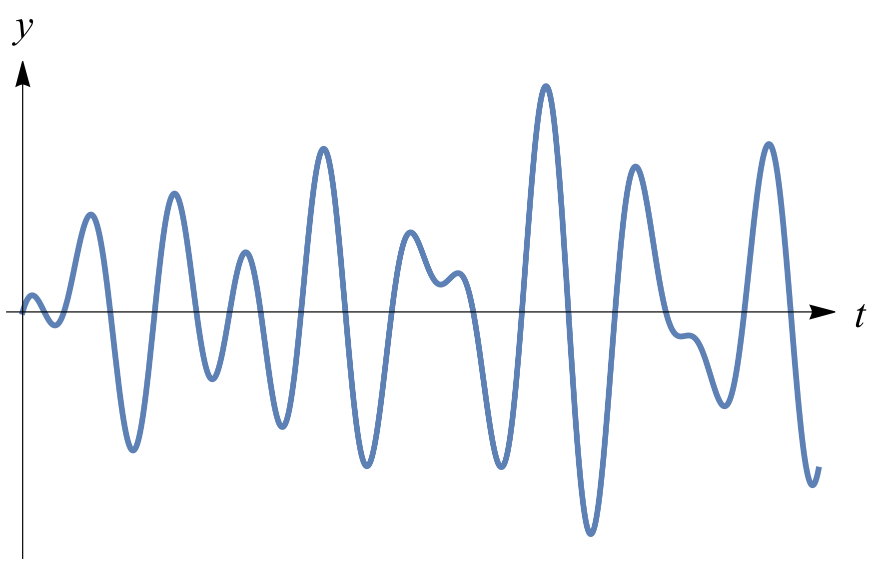
General solution:

$$y = A \sin(\omega t + \phi_0) + \frac{\alpha}{\omega^2 - \gamma^2} \cos \gamma t.$$

ω/γ rational: periodic motion



ω/γ irrational: not periodic



Case 2: $\gamma = \omega$.

$$y'' + \omega^2 y = \alpha \cos \omega t$$

General solution, reduced equation:

$$y = C_1 \cos \omega t + C_2 \sin \omega t$$

$$= A \sin(\omega t + \phi_0).$$

Form of particular solution (undetermined coefficients):

$$z = A\,t \cos \omega t + B\,t \sin \omega t.$$

A particular solution:

$$\frac{\alpha}{2\omega} t \sin \omega t.$$

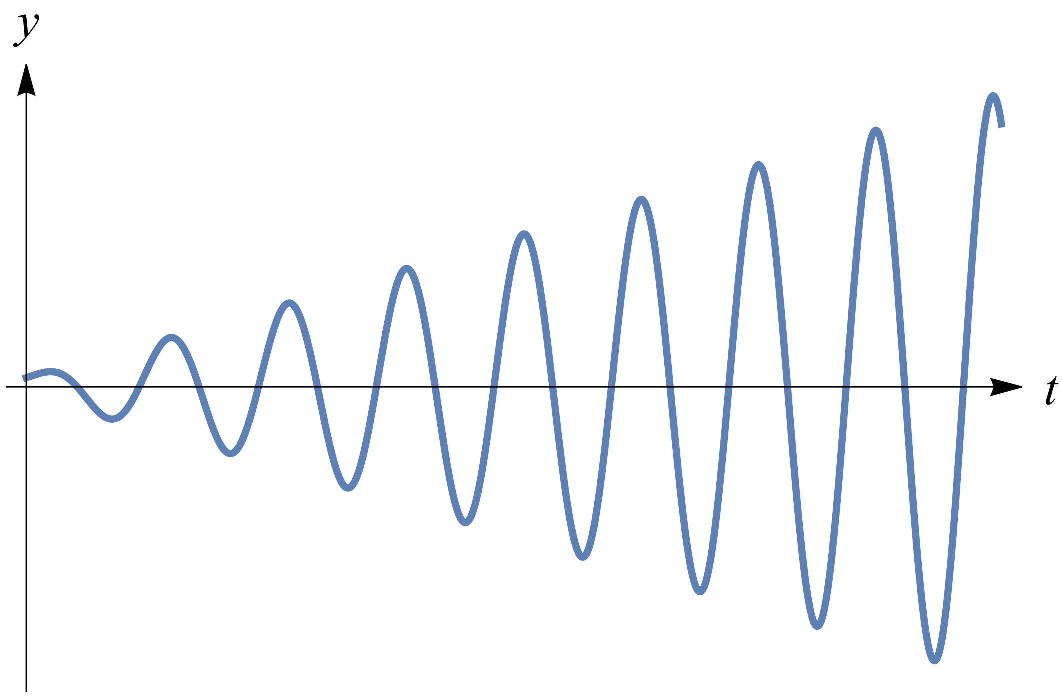
General solution:

$$y = A \sin(\omega t + \phi_0) + \frac{\alpha}{2\omega} t \sin \omega t.$$

Unbounded oscillation

This is known as **resonance**

$$y = A \sin(\omega t + \phi_0) + \frac{\alpha}{2\omega} t \sin \omega t.$$



Never march across a bridge. In April 1831, a brigade of soldiers marched in step across England's Broughton Suspension Bridge. According to accounts of the time, the bridge broke apart beneath the soldiers, throwing dozens of men into the water. After this happened, the British Army reportedly sent new orders: Soldiers crossing a long bridge must "break stride," or not march in unison, to stop such a situation from occurring again. Structures like bridges and buildings, although they appear to be solid and immovable, have a natural frequency of vibration within them. A force that's applied to an object at the same frequency as the object's natural frequency will amplify the vibration of the object in an occurrence called **resonance**. Sometimes your car shakes hard when you hit a certain speed, and a girl on a swing can go higher with little effort just by swinging her legs. The same principle of mechanical resonance that makes these incidents happen also works when people walk in lockstep across a bridge. If soldiers march in unison across the structure, they apply a force at the frequency of their step. If their frequency is closely matched to the bridge's frequency, the soldiers' rhythmic marching will amplify the vibrational frequency of the bridge. If the mechanical resonance is strong enough, the bridge can vibrate until it collapses from the movement. A potent reminder of this was seen in June 2000, when London's Millennium Bridge opened to great fanfare. As crowds packed the bridge, their footfalls made the bridge vibrate slightly. "Many pedestrians fell spontaneously into step with the bridge's vibrations, inadvertently amplifying them," according to a 2005 report in Nature. Though engineers insist the Millennium Bridge was never in danger of collapse, the bridge was closed for about a year while construction crews installed energy-dissipating dampers to minimize the vibration caused by pedestrians

See, also Tacoma Narrows Bridge (Google)

https: *//en.wikipedia/TacomaNarrowsBridge*(1940)

III. Damped Free Vibrations: A resistance force R, called "damping," (e.g., friction) proportional to the velocity $v = y'$ and acting in a direction opposite to the motion:

$$R = -cy' \quad \text{with } c > 0.$$

Force Equation:

$$F = -ky(t) - cy'(t).$$

Newton's Second Law:

$$F = ma = my''$$

Mathematical Model:

$$my''(t) = -ky(t) - cy'(t)$$

or

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \quad (c,\ k,\ m \text{ constant})$$

or

$$y'' + \alpha y' + \beta y = 0 \quad \alpha = \frac{c}{m},\ \beta = \frac{k}{m}$$

$\alpha,\ \beta$ positive constants.

Characteristic equation:

$$r^2 + \alpha r + \beta = 0.$$

Roots

$$r = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

There are three cases to consider:

$$\alpha^2 - 4\beta < 0,$$

$$\alpha^2 - 4\beta > 0,$$

$$\alpha^2 - 4\beta = 0.$$

Case 1: $\alpha^2 - 4\beta < 0$. Complex roots:

(Underdamped)

$$r_1 = -\frac{\alpha}{2} + i\omega, \quad r_2 = -\frac{\alpha}{2} - i\omega$$

where $\omega = \dfrac{\sqrt{4\beta - \alpha^2}}{2}$.

General solution:

$$y = e^{(-\alpha/2)t}\left(C_1 \cos \omega t + C_2 \sin \omega t\right)$$

or

$$y(t) = A\, e^{(-\alpha/2)t} \sin\left(\omega t + \phi_0\right)$$

where

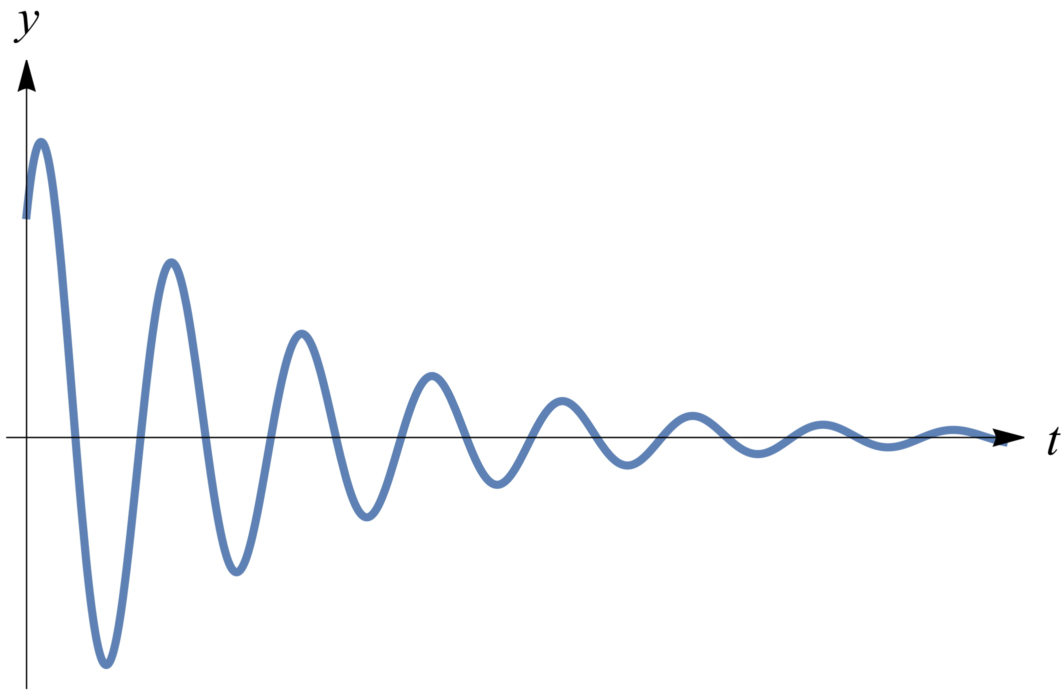
$$A \quad \text{and} \quad \phi_0$$

are constants,

NOTE: The motion is **oscillatory** AND

$$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Underdamped Case:



Case 2: $\alpha^2 - 4\beta > 0$. Two distinct real roots:

(Overdamped)

$$r_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \; r_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.$$

General solution:

$$y(t) = y = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

The motion is nonoscillatory.

NOTE: Since

$$\sqrt{\alpha^2 - 4\beta} < \sqrt{\alpha^2} = \alpha,$$

r_1 and r_2 are **both negative and**

$$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Case 3: $\alpha^2 - 4\beta = 0$. One real root:

(Critically Damped)

$$r_1 = r_2 = \frac{-\alpha}{2},$$

General solution:

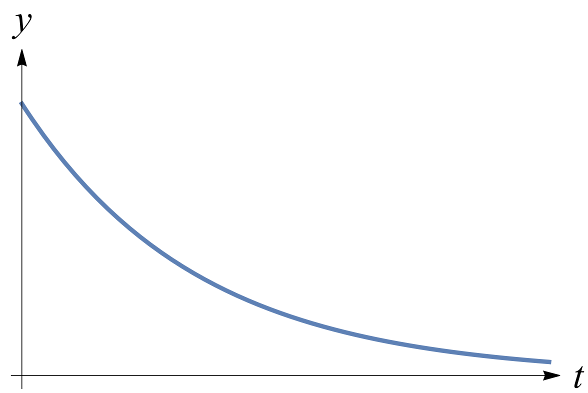
$$y(t) = y = C_1 e^{-(\alpha/2)\,t} + C_2\, t\, e^{-(\alpha/2)\,t}.$$

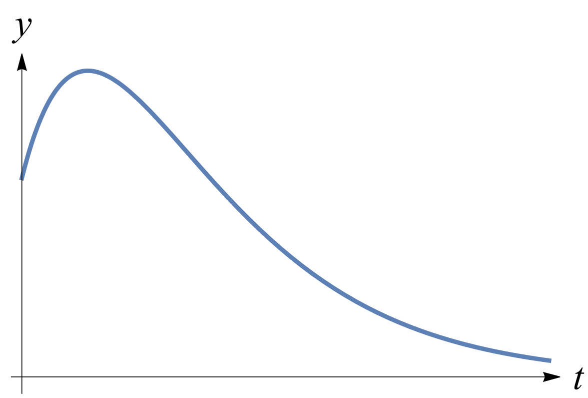
The motion is nonoscillatory and

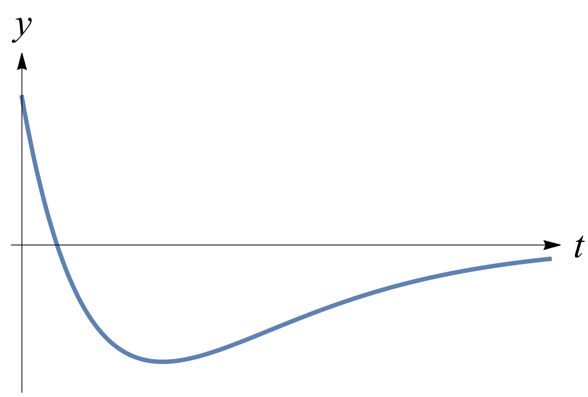
$$y(t) \to 0 \text{ as } t \to \infty.$$

Overdamped and Critically Damped

Cases:







Summary of Case III:

All solutions of

$$y'' + ay' + by = 0$$

have limit 0 as $t \to \infty$.

That is, in the presence of a resistant force (e.g., friction), all solutions ultimately return to the equilibrium position.

IV. Forced Damped Vibrations

Apply an external force G to a damped, freely vibrating system

Force Equation:

$$F = -ky - cy' + G.$$

Mathematical Model:

$$my'' = -ky - cy' + G$$

or
$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{G}{m},$$

which we write as

$$y'' + \alpha y' + \beta y = g,$$

where $\alpha = c/m,\ \beta = k/m,\ g = G/m$

A periodic external force:

$$g = a \cos \gamma t, \quad a,\ \gamma > 0 \text{ const.}$$

Mathematical Model:

$$y'' + \alpha y' + \beta y = a \cos \gamma t$$

General solution:

$$y(t) = C_1\, y_1(t) + C_2\, y_2(t) + Z(t)$$

$$= Y_c(t) + Z(t),$$

Note: From Case III

$$\lim_{t \to \infty} Y_c(t) = 0.$$

as $t \to \infty$ so

$$\lim_{t \to \infty} y(t) = Z(t).$$

Particular solution of

$$(N) \quad y'' + \alpha\, y' + \beta\, y = a \cos \gamma t$$

will have the form:

$$Z(t) = A \cos \gamma t + B \sin \gamma t.$$

General solution of (N):

$$y(t) = Y_c(t) + Z(t)$$

Note: $\lim_{t\to\infty} y(t) = Z(t)$

$Y_c(t)$, the general solution of the reduced equation, is called the **transient solution**.

$Z(t)$ a particular solution of (N), is called a **steady state solution**.

Example 1: $y'' + \frac{3}{4}y' + \frac{1}{8}y = \cos t$

General solution

$$y = C_1 e^{-t/4} + C_2 e^{-t/2} + \frac{56}{85}\cos t - \frac{48}{85}\sin t$$

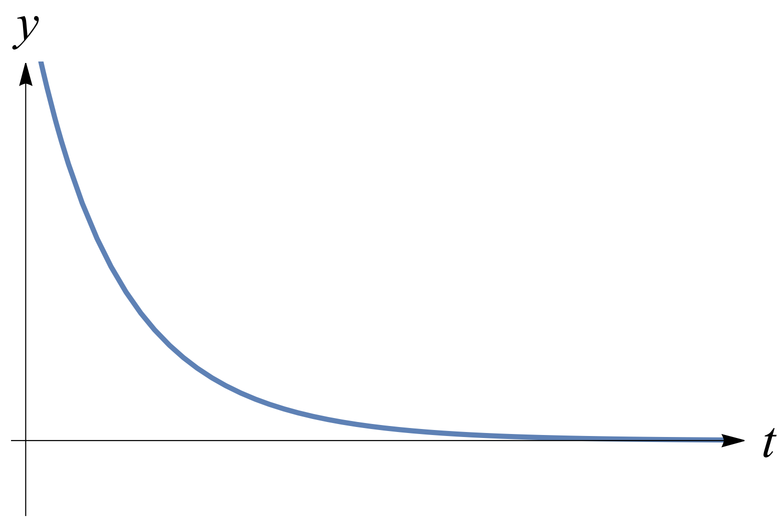
Transient solution:

$$y(t) = 2e^{-t/4} + e^{-t/2} \quad (C_1 = 2,\ C_2 = 1)$$

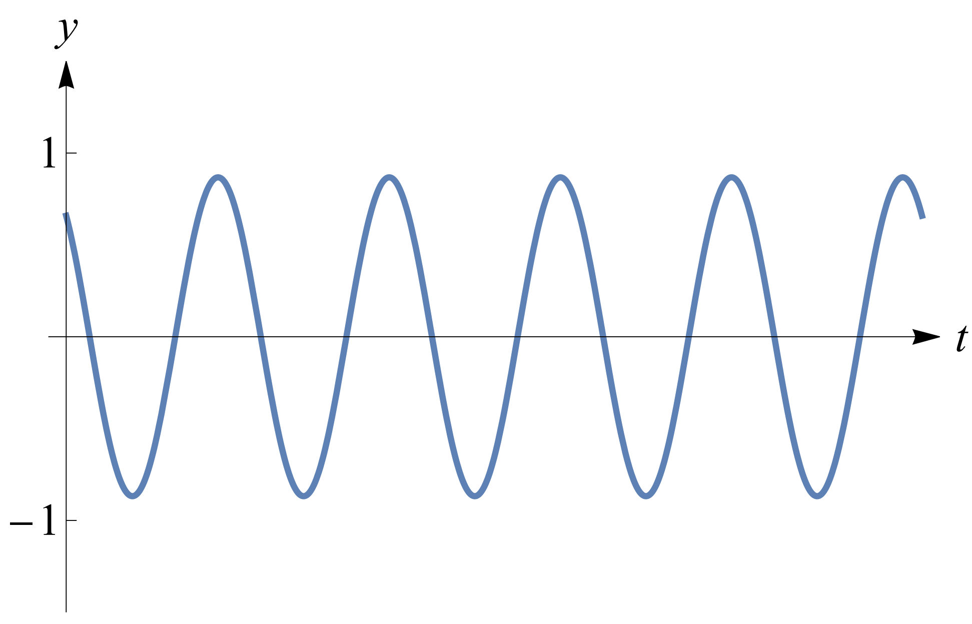
Steady-state solution:

$$Z(t) = \frac{56}{85}\cos t - \frac{48}{85}\sin t$$

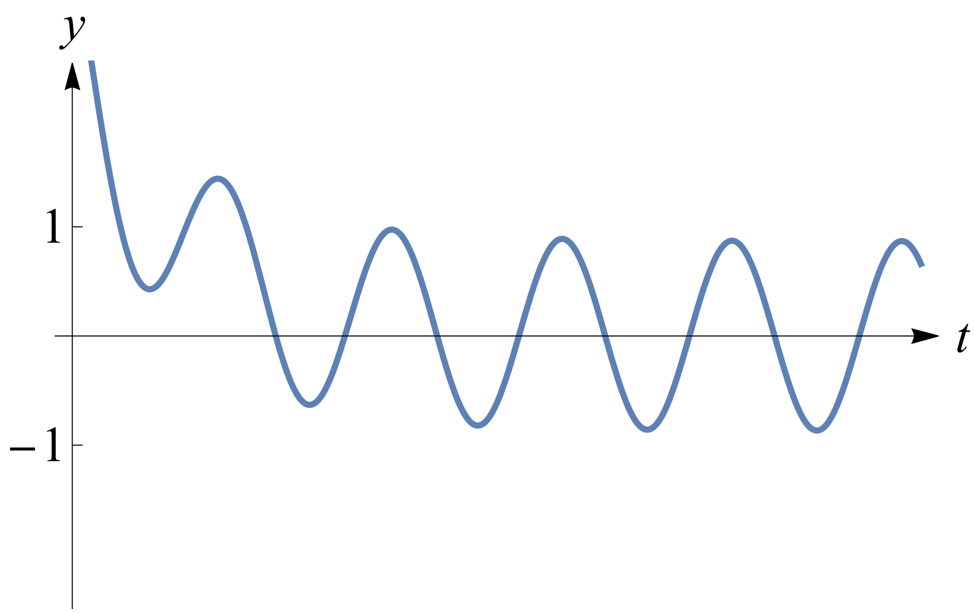
Transient solution:



Steady-state solution:



$$y = 2e^{-t/4} + e^{-t/2} + \frac{56}{85}\cos t - \frac{48}{85}\sin t$$



Example 2: $y'' + 2y' + 5y = \cos t$

General solution

$$y = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t +$$

$$\frac{1}{5} \cos t + \frac{1}{10} \sin t$$

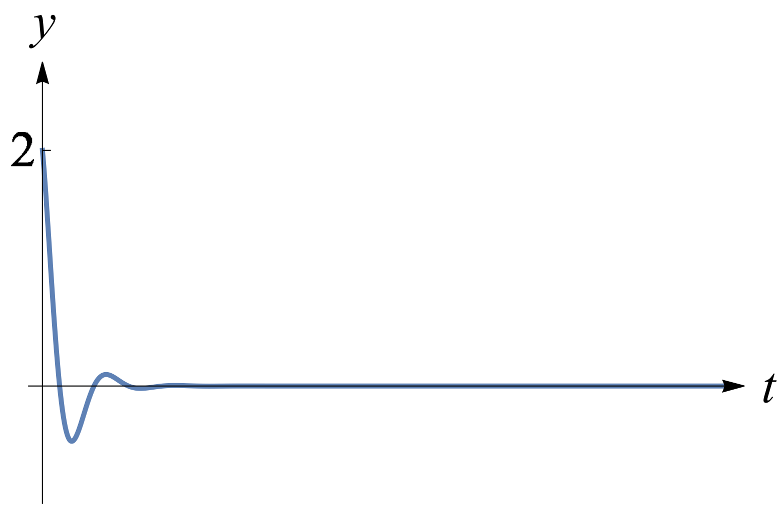
A transient solution:

$$y(t) = 2e^{-t} \cos 2t \quad (C_1 = 2,\ C_2 = 0)$$

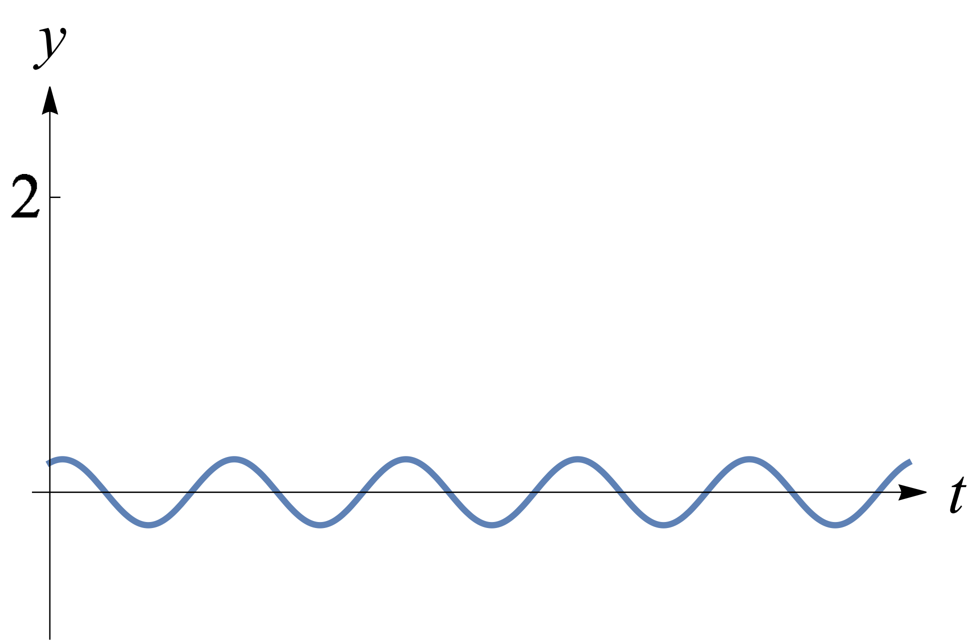
Steady-State solution:

$$Z(t) = \frac{1}{5} \cos t + \frac{1}{10} \sin t$$

Transient solution: $Y(t) = 2e^{-t}\cos 2t$



Steady-state solution: $Z(t) = \frac{1}{5}\cos t + \frac{1}{10}\sin t$



$$y = 2e^{-t}\cos 2t + \frac{1}{5}\cos t + \frac{1}{10}\sin t$$